

Mirror matter and the self-duality of $S(U(2) \times U(3))$

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We give an explicit root-datum isomorphism showing that the Standard Model gauge group $G_{\text{SM}} = S(U(2) \times U(3)) = (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$ is Langlands self-dual. Among the quotients by subgroups of $\mathbb{Z}_6$, only the $\mathbb{Z}_6$ quotient has this property. On the product theory $G_{\text{SM}} \times G'_{\text{SM}}$ we define a mirror duality $\mathcal{M}_P = \text{swap} \circ P \circ (S \otimes S)$, with S treated as a formal map, since S -duality is expected to be exact only in the $\mathcal{N} = 4$ theory. $\mathcal{M}_P$ fixes the couplings on the locus $\tau' = 1/\bar{\tau}$, which at $\theta = 0$ gives $g_3 g'_3 = g_2 g'_2 = 4\pi$ and $g_Y g'_Y = 12\pi$ ($g_1 g'_1 = 20\pi$ in grand-unified normalization). The construction does not require a self-dual group; the self-duality makes the electric-frame group of the mirror sector equal to G_{SM} , at the price of a charge-conjugation twist in the dictionary. If per-sector S -duality is an exact equivalence, $\mathcal{M}_P$ is the standard mirror parity written in a dual frame of the mirror sector; if it is not, as the running couplings and the $B + L$ anomaly of the chiral Standard Model indicate, $\mathcal{M}_P$ is not a symmetry. The duality maps ordinary Wilson lines to mirror 't Hooft lines; for example, the electron's line goes to the minimal pure-hypercharge monopole line of the mirror sector. In the mirror's S -dual frame, $\mathcal{M}_P$ forces the mixed θ -angle of that frame to vanish but leaves the kinetic mixing in that frame free. We work out the consequences for mirror-matter phenomenology: the running of the couplings, the selection rules on the operators that couple the two sectors, and neutron-mirror-neutron oscillations. On the Hitchin moduli spaces of the two sectors, the duality acts, at the level of lattices, as the exchange of the sectors composed with fiberwise T -duality.

I. INTRODUCTION

The idea that each ordinary particle has a mirror partner goes back to the discussion of parity nonconservation by Lee and Yang [1], as recounted in Ref. [2]; the mirror-particle sector was developed by Kobzarev, Okun and Pomeranchuk [3], and Refs. [2, 4] review its history. In its usual form the gauge group is $G \times G'$, with an ordinary and a mirror sector that share spacetime and gravity but no gauge interaction, and the theory is invariant under an exchange of the two sectors combined with parity [4, 5]. In the mirror-matter theory of Refs. [6–10] both factors are the Standard Model gauge group, $G = G' = G_{\text{SM}}$, the two sectors interact only through gravity and through oscillations of neutral hadrons into their mirror partners, of which n – n' and K^0 – $K^{0'}$ are the most significant, and the mirror symmetry is broken spontaneously by a slight difference of the two Higgs vacuum expectation values, $(v' - v)/v \sim 10^{-15}$ – 10^{-14} , a rough range for the model in general. For n – n' , with experimental constraints, the neutron and mirror-neutron masses are split by $\Delta m_n \simeq 3$ – $12 \mu\text{eV}$ [9], with a small mixing angle, so that Δm_n is also the splitting of the mass eigenstates and the oscillations are in the nondegenerate regime. These oscillations have been proposed as the origin of the neutron lifetime anomaly and of dark matter [6, 9], and of the baryon asymmetry of the universe [7]. In Ref. [10] the mirror map is an orientation symmetry of internal space, the exchange of the sectors combined with a chirality operation, and it is related to T -duality and to Calabi–Yau

mirror symmetry in two-dimensional models. None of these works relates the mirror map to electric-magnetic duality in four dimensions.

The Lie algebra and the representations of the Standard Model fix its gauge group only up to a finite quotient; with the observed matter the group can be $(SU(3) \times SU(2) \times U(1))/\Gamma$ with $\Gamma \in \{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6\}$ [11, 12], and the global form fixes the spectrum of line operators [13]. For $\Gamma = \mathbb{Z}_6$ the group is $S(U(2) \times U(3))$ [14]. In $\mathcal{N} = 4$ super-Yang–Mills theory, Montonen–Olive duality, or S -duality [15–19], is conjectured to relate the theory with gauge group G and coupling τ to the theory with the Goddard–Nuyts–Olive (GNO), or Langlands, dual group $G^\vee$ and coupling $-1/\tau$ [19–22], and to exchange electric and magnetic charges; for the chiral, non-supersymmetric Standard Model no such duality is established. The dual group is the group whose electric charges are the magnetic charges of G , and G is *Langlands self-dual* if $G^\vee \cong G$: its electric and magnetic charges then have the same structure, and S maps a theory with gauge group G to a theory with the same gauge group. This suggests the question that motivates the present work: since $S(U(2) \times U(3))$ turns out to be isomorphic to its Langlands dual, can S -duality of G_{SM} be used to construct the map between the ordinary and the mirror sectors?

S -duality alone does not provide such a map. It is an equivalence between two descriptions of one theory; since $G_{\text{SM}}^\vee \cong G_{\text{SM}}$ it maps the family of G_{SM} theories to itself, but it is a symmetry of a single theory only at a self-dual coupling, and it cannot produce a second, independent sector. We therefore consider the product theory $T[G_{\text{SM}}, \tau] \otimes T[G'_{\text{SM}}, \tau']$ of an ordinary and a mirror sector and define two maps on it. The first, $\mathcal{M} = \text{swap} \circ (S \otimes S)$,

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replaces each sector by its S-dual description and exchanges the two; the second, $\mathcal{M}_P = \text{swap} \circ P \circ (S \otimes S)$, also reverses the orientation of spacetime, as the mirror parity of conventional mirror matter does, and is the one that acts on chiral matter. Since S changes which charges are called electric, statements about couplings and charges refer to a description, or frame: the original frame, in which each sector uses its own electric variables, or the S-dual frame of the mirror sector, in which the mirror sector is described by the dual variables. Throughout, S is treated as a formal map on couplings, charge lattices and line operators: we do not assume that the chiral Standard Model is S-dual, and the conditions under which S can hold for G_{SM} are collected in App. B. Accordingly, when we say that $\mathcal{M}_P$ fixes the couplings on a locus, we mean that its formal action maps every point of that locus to itself.

Our answer to the question is a qualified one. The maps $\mathcal{M}$ and $\mathcal{M}_P$ do not require a self-dual group: for any G they act on $T[G, \tau] \otimes T[G^\vee, \tau']$, and in the S-dual frame of the second factor this product is two copies of $T[G]$, i.e. conventional mirror matter. What the self-duality of G_{SM} adds comes down to three things: the electric-frame group of the mirror sector is again G_{SM} , both sectors share one lattice of θ periods, and the dictionary carries a charge-conjugation twist that cannot be removed, the price of identifying $G_{\text{SM}}^\vee$ with G_{SM} . A single G_{SM} sector also has a self-dual point, $g_3 = g_2 = \sqrt{4\pi}$ and $g_Y = \sqrt{12\pi}$ at $\theta = 0$, under an S-type map of all three factors (S followed by an identification of roots with co-roots). For every global form the matter alone forces the mirror matter to be magnetic; for the non-self-dual forms the gauge group forces it too. The physical status of $\mathcal{M}_P$ then rests on S , and there are two cases. If per-sector S is an exact equivalence, then in the S-dual frame of the mirror sector, $\mathcal{M}_P$ is the standard mirror parity of Refs. [3–5], $\text{swap} \circ (P \otimes P)$: the exchange of the sectors combined with spacetime parity, written in an electric frame for the ordinary sector and a magnetic frame for the mirror sector. It is the natural choice, since it reverses the orientation of both spacetime and the internal space. Its fixed locus maps onto equal couplings in that frame, so the theory has the spectrum and observables of conventional mirror matter, and relations such as $g_3 g'_3 = 4\pi$ state in which frame the mirror sector is described. If S is not an exact equivalence, as the running couplings and the $B + L$ anomaly indicate for the chiral Standard Model, $\mathcal{M}_P$ is not a symmetry, and its coupling relations can hold at most at one ultraviolet scale, where they are a boundary condition rather than the consequence of a symmetry. Consequently, either the mirror neutron is sufficiently heavy to render n – n' oscillations negligible, or a mass degeneracy within a few μeV necessitates either extreme fine-tuning of inter-sector couplings or—more plausibly, by analogy with neutrino oscillations—an origin in non-perturbative vacuum mismatch and/or topological mechanisms from unexplored dimensions.

The construction therefore does not derive the mirror sector from the self-duality, and it does not predict mirror couplings that are independent of a choice of frame. What it provides are constraints that do not depend on the frame, collected in Sec. IIIH: the uniqueness of the $\mathbb{Z}_6$ global form, a necessary condition on the matter for $\mathcal{M}_P$ to be a symmetry, the orientation character that distinguishes mirror from copy matter, and statements on when kinetic mixing vanishes exactly. Where $\mathcal{M}_P$ is a symmetry it acts, in the S-dual frame, as the standard mirror parity, and its selection rules on the operators that couple the sectors are those of any model with exact mirror parity [4, 5, 23]. We classify these operators completely up to dimension 6, with the six-quark operators that mix n and n' : $\mathcal{M}_P$ forbids exactly the mixed θ -angle of the S-dual frame and a dimension-6 neutrino portal, and it ties every other portal to its mirror image. For n and n' it forces equal flavor-basis masses and a mixing angle $\pi/4$ for any mixing mass; this is the degenerate regime of the storage experiments, which bounds that mass by $\hbar/\tau_{nn'}$, so that the nondegenerate regime above needs $\mathcal{M}_P$ to be broken, e.g. by $v' \neq v$. We do not address the dynamics that would make S exact, the origin of the n – n' mixing angle, or the Calabi–Yau threefold of the compactification of Ref. [10].

The paper is organized as follows. Section II establishes the self-duality through an explicit isomorphism F of the root datum of G_{SM} onto its dual, shows that among the quotients by subgroups of $\mathbb{Z}_6$ only the $\mathbb{Z}_6$ quotient is self-dual, and states the criterion for $S(U(n_1) \times \dots \times U(n_k))$. Section III defines $\mathcal{M}$ and $\mathcal{M}_P$, derives their fixed loci, $g_3 g'_3 = g_2 g'_2 = 4\pi$ and $g_Y g'_Y = 12\pi$ at $\theta = 0$ ($g_1 g'_1 = 20\pi$ in grand-unified normalization), shows that the helicity flip between the sectors comes from parity and not from the twist, isolates what self-duality adds, and separates frame-independent from frame-dependent statements. Section IV maps line operators and generalized symmetries: ordinary Wilson lines go to mirror 't Hooft lines, the electron's line, for example, to the minimal pure-hypercharge monopole line of the mirror sector. Section V treats couplings, the portal operators between the sectors, kinetic mixing and n – n' phenomenology. Section VI gives the counterpart on the Hitchin moduli spaces [19, 24, 25]: at the level of zero-mode lattices on generic fibers, $\mathcal{M}$ acts as the exchange of the sectors composed with fiberwise T-duality, and its abelian part is the compact-boson T-duality of the two-dimensional mirror model SMM2b of Ref. [10]. Section VII contains our conclusions. App. A collects our conventions, App. B the conditions under which S can hold for G_{SM} , App. C a two-dimensional model with charged supersymmetry, and App. D the proof of the self-duality criteria.

II. LANGLANDS SELF-DUALITY OF G_{SM}

The Lie algebra and the matter content do not fix the global form of the Standard Model gauge group: with the observed representations it can be $(SU(3) \times SU(2) \times U(1))/\Gamma$ with $\Gamma \in \{1, \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_6\}$ [11, 12]. For $\Gamma = \mathbb{Z}_6$ the group is $S(U(2) \times U(3))$, the subgroup of block-diagonal matrices in $SU(5)$ [14]. The global form fixes the charges of the Wilson lines; the magnetic charges of the 't Hooft lines are weights of the Goddard–Nuyts–Olive (GNO) dual group [13, 20, 26, 27], which is the Langlands dual group [19, 21, 22]. In this section we show that $G_{\text{SM}} \equiv S(U(2) \times U(3))$ is isomorphic to its Langlands dual, give the isomorphism explicitly at the level of root data, of groups, and of charge lattices, and determine which groups of the same type share this property. For mirror matter this is the starting point: S-duality maps a G_{SM} theory to a $G_{\text{SM}}^\vee$ theory, and only if the two groups are isomorphic can the dual description of a sector be read as a Standard Model sector again. The explicit isomorphism also fixes the dictionary between the two descriptions, which the mirror maps of Sec. III use, and its charge-conjugation twist (Sec. IID).

We work with the complex groups $S(GL_2 \times GL_3)$ and their relatives; a compact connected Lie group and its complexification have the same root datum, and by the isomorphism and existence theorems connected reductive groups are classified up to isomorphism by their root data [28, 29]. Every statement below therefore holds for both forms.

A. Root datum and an explicit self-duality

A root datum $\mathcal{R} = (X^*, \Phi, X_*, \Phi^\vee)$ consists of the character lattice X^* and the cocharacter lattice X_* of a maximal torus, in perfect duality, together with the roots $\Phi \subset X^*$ and the coroots $\Phi^\vee \subset X_*$. Its dual $\mathcal{R}^\vee = (X_*, \Phi^\vee, X^*, \Phi)$ is again a root datum [28, 29], and the Langlands dual group $G^\vee$ is the group whose root datum is $\mathcal{R}^\vee$ [22, 29]. A group is Langlands self-dual if $\mathcal{R}$ and $\mathcal{R}^\vee$ are isomorphic.

For $G = S(GL_2 \times GL_3)$ we take the maximal torus $T = \{(\text{diag}(t_1, t_2), \text{diag}(t_3, t_4, t_5)) : t_1 \cdots t_5 = 1\}$. Its lattices are

$$X^* = \mathbb{Z}^5 / \mathbb{Z}(1, 1, 1, 1, 1), \quad X_* = \left\{ \lambda \in \mathbb{Z}^5 : \sum_i \lambda_i = 0 \right\}, \quad (1)$$

with the pairing induced by the dot product and the dual bases $\chi_i = [u_i]$ and $\lambda_i = u_i - u_5$ ($i = 1, \dots, 4$), where u_i is the standard basis of $\mathbb{Z}^5$. The roots are the eight vectors $u_i - u_j$ with $i \neq j$ both in the first block $\{1, 2\}$ or both in the second block $\{3, 4, 5\}$; the coroots are the same vectors, read in X_* . The Weyl group $W = S_2 \times S_3$ has order 12, and $X^*/\mathbb{Z}\Phi \cong X_*/\mathbb{Z}\Phi^\vee \cong \mathbb{Z}$: the center of G is a one-dimensional torus and $\pi_1(G) \cong \mathbb{Z}$, both without torsion.

In these bases the self-duality is the matrix

$$F = \begin{pmatrix} -1 & -2 & 1 & 1 \\ -2 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix} : X^* \rightarrow X_*, \quad \det F = -1. \quad (2)$$

It maps every root to its coroot, $F\alpha = \alpha^\vee$, while $F^\vee \equiv (F^T)^{-1}$ maps every coroot back to its root, $F^\vee \alpha^\vee = \alpha$; it preserves the pairing, $\langle F^\vee \lambda, Fx \rangle = \langle x, \lambda \rangle$, and intertwines the Weyl groups, $FwF^{-1} = (w^T)^{-1}$ for all $w \in W$. The pair $(F, F^\vee)$ is therefore an isomorphism of root data $\mathcal{R} \rightarrow \mathcal{R}^\vee$, and $S(GL_2 \times GL_3)$, as well as its compact form G_{SM} , is isomorphic to its Langlands dual.

Since F is symmetric, the self-duality can also be stated as a property of a bilinear form. Lifted to $\mathbb{Z}^5$, $B(x, y) = x^T F y$ reads

$$B(x, y) = x \cdot y - X_2 Y_2 - (X_2 - X_3)(Y_2 - Y_3), \quad (3)$$

where $X_2 = x_1 + x_2$ and $X_3 = x_3 + x_4 + x_5$ are the block traces (and likewise Y_2, Y_3 for y); it vanishes on $(1, 1, 1, 1, 1)$ and so is defined on X^* . The form B is W -invariant, unimodular and odd, gives every root norm 2, and has signature $(3, 1)$. The self-duality is not unique: there are $|\text{Isom}(\mathcal{R}, \mathcal{R}^\vee)| = |\text{Aut } \mathcal{R}| = 24$ isomorphisms, with $\text{Aut } \mathcal{R} = W \times \{\pm 1\}$. The count has a simple origin. Up to W , an isomorphism $\mathcal{R} \rightarrow \mathcal{R}^\vee$ is fixed by its signs ϵ_2, ϵ_3 on the $SU(2)$ and $SU(3)$ roots and its sign η on the central directions, and it is compatible with the way the two blocks are glued in X^* and X_* if and only if $3 \equiv \eta \epsilon_2 \pmod{2}$ and $2 \equiv \eta \epsilon_3 \pmod{3}$ (App. D). The first condition is empty, and $\epsilon_2 = -1$ is a Weyl reflection; the second leaves $\eta \epsilon_3 = -1$, i.e. the two classes $\pm F W$, each with $|W| = 12$ elements. Exactly four of them are W -invariant symmetric forms; they differ by the signs ϵ_2, ϵ_3 with which they act on the $SU(2)$ and $SU(3)$ roots, have signatures $(3, 1), (2, 2), (2, 2)$ and $(1, 3)$, and none of them is definite.

B. The quotient presentations

The Langlands dual of $S(GL_2 \times GL_3)$ can be identified directly. For a reductive group G and a central torus Z_ν with primitive cocharacter ν , the root data of the quotient G/Z_ν and of the kernel of a central character χ are

$$\begin{aligned} D(G/Z_\nu) &= (X^* \cap \nu^\perp, \Phi, X_*/\mathbb{Z}\nu, \Phi^\vee), \\ D(\ker \chi) &= (X^*/\mathbb{Z}\chi, \Phi, X_* \cap \chi^\perp, \Phi^\vee), \end{aligned} \quad (4)$$

so that $(G/Z_\nu)^\vee = \ker(\chi_\nu : G^\vee \rightarrow \mathbb{G}_m)$ with $\chi_\nu = \nu$: Langlands duality exchanges the quotient by a central torus with the kernel of a central character. For the self-dual group $GL_2 \times GL_3$ [22, 28], with $\nu = (1, 1, 1, 1, 1)$ and $\chi_\nu = \det A \det B$, this gives $((GL_2 \times GL_3)/\mathbb{G}_m)^\vee = S(GL_2 \times GL_3)$, where $\mathbb{G}_m$ is the diagonal subgroup of scalars (t_{12}, t_{13}) . The self-duality of Sec. IIA is thus

the statement that the quotient and the subgroup are isomorphic groups. An explicit isomorphism is

$$\Psi : (GL_2 \times GL_3)/\mathbb{G}_m \rightarrow S(GL_2 \times GL_3),$$

$$[(A, B)] \mapsto \left(A \frac{\det B}{(\det A)^2}, B \frac{\det A}{\det B} \right), \quad (5)$$

with inverse $\Sigma(A', B') = [(A', \det A' B')]$; the kernel of Ψ on $GL_2 \times GL_3$ is exactly $\mathbb{G}_m$. The naive map, the inclusion of $S(GL_2 \times GL_3)$ in $GL_2 \times GL_3$ followed by the projection, is onto but has kernel μ_5 . On cocharacters, in the bases of Sec. II A, Ψ induces exactly the matrix of Eq. (2), $\Psi_* = F$, and $\Sigma_* = F^{-1}$, while the naive map has $|\det| = 5$. The lattice self-duality is therefore the shadow of the group isomorphism Ψ .

The compact forms carry the same structure. With $z = e^{i\vartheta}$ acting on a field of hypercharge Y as $e^{6iY\vartheta}$, so that $Y(Q_L) = \frac{1}{6}$ and $Q = T_3 + Y$,

$$\varphi : SU(3) \times SU(2) \times U(1)_Y \rightarrow S(U(2) \times U(3)),$$

$$(h, g, z) \mapsto (z^3 g, z^{-2} h), \quad (6)$$

is surjective with kernel $\{(z^2 1_3, z^{-3} 1_2, z) : z^6 = 1\} \cong \mathbb{Z}_6$, generated by $(e^{2\pi i/3} 1_3, -1_2, e^{i\pi/3})$; hence $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6 \cong S(U(2) \times U(3))$, as in Ref. [14] up to the normalization of the $U(1)$ charge. On $\mathbb{C}^2 \oplus \mathbb{C}^3$ the hypercharge is $Y = \text{diag}(\frac{1}{2}, \frac{1}{2}, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$, and one generation consists of the representations $Q_L = A \otimes B$, $u_R = \det A B$, $d_R = B$, $L_L = (\det A)^{-1} A$, $e_R = (\det A)^{-1}$, and the Higgs doublet is $H = A$, for $(A, B) \in S(U(2) \times U(3))$. The maps Ψ and Σ restrict to an isomorphism $(U(2) \times U(3))/U(1) \cong S(U(2) \times U(3))$, with $U(1)$ the unitary scalars. In this quotient presentation the hypercharge group is $U(1)_Y = \{(z^{-1} 1_2, 1_3)\} = \{[(1_2, z 1_3)]\}$, and $\Psi(z^{-1} g, h) = \varphi(h, g, z)$. The naive inclusion $S(U(2) \times U(3)) \rightarrow (U(2) \times U(3))/U(1)$ again has kernel $\mathbb{Z}_5$.

G_{SM} thus has three equivalent presentations: as the subgroup $S(U(2) \times U(3))$, as the finite quotient $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$, and as the quotient $(U(2) \times U(3))/U(1)$ by a central circle, which by Eq. (4) is the presentation of its Langlands dual.

C. Charge lattices

In physical terms, the character lattice is the lattice of electric charges and the cocharacter lattice that of magnetic charges. We label a weight of G_{SM} by (a_1, a_2, n, y) , with (a_1, a_2) the $SU(3)$ Dynkin labels, n the $SU(2)$ Dynkin label ($n = 2j$), and $y = 6Y$. The electric and magnetic lattices are

$$\Lambda_e = \{(a_1, a_2, n, y) \in \mathbb{Z}^4 :$$

$$2(a_1 + 2a_2) + 3n + y \equiv 0 \pmod{6}\},$$

$$\Lambda_m = \{(b_1, b_2, n^\vee, \tilde{Y}) \in \mathbb{Z}^4 : \tilde{Y} \equiv n^\vee \pmod{2},$$

$$\tilde{Y} + b_1 + 2b_2 \equiv 0 \pmod{3}\}, \quad (7)$$

TABLE I. Highest weights (a_1, a_2, n, y) of one Standard Model generation and of the Higgs doublet in Λ_e , and their images $(b_1, b_2, n^\vee, \tilde{Y})$ in Λ_m under the self-duality (9).

Field	Y	(a_1, a_2, n, y)	$(b_1, b_2, n^\vee, \tilde{Y})$
Q_L	$\frac{1}{6}$	$(1, 0, 1, 1)$	$(1, 0, 1, -1)$
u_R	$\frac{2}{3}$	$(1, 0, 0, 4)$	$(1, 0, 0, -4)$
d_R	$-\frac{1}{3}$	$(1, 0, 0, -2)$	$(1, 0, 0, 2)$
L_L	$-\frac{1}{2}$	$(0, 0, 1, -3)$	$(0, 0, 1, 3)$
e_R	-1	$(0, 0, 0, -6)$	$(0, 0, 0, 6)$
H	$\frac{1}{2}$	$(0, 0, 1, 3)$	$(0, 0, 1, -3)$

and $\Lambda_m = \Lambda_e^*$ is the Dirac dual of Λ_e under the pairing

$$\langle e, m \rangle = a^T A_3^{-1} b + \frac{1}{2} n n^\vee + Y \tilde{Y}, \quad (8)$$

where A_3 is the Cartan matrix of $SU(3)$. The congruence in Λ_e is the condition that the generator of $\mathbb{Z}_6$ act trivially; it holds for all Standard Model fields. The magnetic coordinate $\tilde{Y}$ is defined lattice-theoretically; its overall sign depends on the orientation convention for monopoles. In these coordinates the self-duality of Eq. (2) is the map

$$(a_1, a_2, n, Y) \mapsto (b_1, b_2, n^\vee, \tilde{Y}) = (a_1, a_2, n, -6Y), \quad (9)$$

which maps Λ_e onto Λ_m . The four W -invariant choices of Sec. II A are $(a, n, y) \mapsto (\epsilon_3 a, \epsilon_2 n, -\epsilon_3 y)$, and the invariant form is $B = a^T A_3^{-1} a' + \frac{1}{2} n n' - 6Y Y'$.

Table I lists the magnetic duals of the Standard Model fields. The dual of Q_L is a minimal monopole, in the sense that it has the smallest nonzero $|\tilde{Y}| = 1$ on Λ_m ; it is not the only one, since $(0, 1, 1, 1)$ also has $|\tilde{Y}| = 1$. The dual of e_R is the minimal pure-hypercharge monopole: with $b = 0$ and $n^\vee = 0$ the smallest value is $|\tilde{Y}| = 6$.

The self-duality singles out the $\mathbb{Z}_6$ quotient. Among the quotients $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_k$ by the subgroups $\mathbb{Z}_k$, $k \mid 6$, of the $\mathbb{Z}_6$ above, with $Y(Q_L) = \frac{1}{6}$ held fixed, the groups with $k = 1, 2, 3$ are not self-dual: their centers have component groups $\mathbb{Z}_6, \mathbb{Z}_3$ and $\mathbb{Z}_2$, while their fundamental groups are torsion free, so that $X^*/\mathbb{Z}\Phi$ and $X_*/\mathbb{Z}\Phi^\vee$ cannot be isomorphic. Only $k = 6$ is self-dual.

D. The dictionary on the Cartan algebra

The isomorphism Ψ of Eq. (5) relates the two presentations of G_{SM} in a way that is not the identity on matrices. Let $\hat{G} \equiv (U(2) \times U(3))/U(1)$ and let $\iota : \mathfrak{su}(\mathfrak{u}_2 \oplus \mathfrak{u}_3) \rightarrow (\mathfrak{u}_2 \oplus \mathfrak{u}_3)/\mathfrak{u}(1)$ be the identity on matrices. Then

$$d\Psi(\xi, \eta) = (\xi + (\text{tr } \eta - 2 \text{tr } \xi) 1_2,$$

$$\eta + (\text{tr } \xi - \text{tr } \eta) 1_3), \quad (10)$$

$$\mathbf{M} \equiv d\Psi \circ \iota = 1 - 6P_y,$$

where P_y is the projection on the hypercharge direction $y = \text{diag}(3, 3, -2, -2, -2) = 6Y$. The map $\mathbf{M}$ is the identity on $\mathfrak{su}(2) \oplus \mathfrak{su}(3)$ and multiplies the hypercharge by

-5 , $My = -5y$; on the group, it acts on the central torus $(z^3 1_2, z^{-2} 1_3)$ as $z \mapsto z^{-5}$. The factor 5 is intrinsic: $\iota(y) = 5\nu$, with $\nu = [(1, 1, 0, 0)]$ the primitive central cocharacter of $\hat{G}$, and $5 = |S(U(2) \times U(3)) \cap U(1)_{\text{diag}}| = m + n$ for $(m, n) = (2, 3)$; all y -charges of representations of $\hat{G}$ lie in $5\mathbb{Z}$. The sign is fixed by the $\mathbb{Z}_6$ glue: the centers of $\hat{G}$ and of $S(U(2) \times U(3))$ are glued to the centers of $SU(2) \times SU(3)$ in opposite ways. When $U(1)_Y$ is transported to $\hat{G}$ as in Sec. IIB, Ψ is the identity on Standard Model labels; the -5 refers to the matrix identification ι .

The same factor appears in the matrix F . Let K be the Gram matrix of the trace form $\text{tr}(\lambda\lambda')$ on X_* ; it is the matrix of ι_* (the naive map of Sec. IIB), with $\det K = 5$. Then

$$FK = M, \quad V^{-1}FKV = \text{diag}(1, 1, 1, -5), \quad (11)$$

in the basis $(\alpha^\vee, \beta_1^\vee, \beta_2^\vee, y)$ of an $SU(2)$ coroot, two $SU(3)$ simple coroots and the hypercharge direction. Equivalently, the form $B^\vee = F^{-1}$ on X_* is the trace form on the coroots and $-\frac{1}{5}$ of it on the hypercharge line, and in the coordinates of Eq. (9), $y(\iota\lambda) = 5\tilde{Y}(\lambda)$. The factor -5 is thus the sign of the $\mathbb{Z}_6$ glue twist times the ratio $5 = m + n$ of the trace normalization of $SU(5)$ to the lattice normalization of the hypercharge.

The sign can be moved from the hypercharge to color. The C-twisted dictionary

$$\begin{aligned} \Psi_C &\equiv \text{Ad}(\sigma_2 \times 1) \circ c \circ \Psi : \\ [(A, B)] &\mapsto \left(A \frac{\det A}{\det B}, (B^T)^{-1} \frac{\det B}{\det A} \right), \end{aligned} \quad (12)$$

where c is complex conjugation, is again an isomorphism $\hat{G} \rightarrow S(U(2) \times U(3))$, with inverse $(A', B') \mapsto [(A', (B'^T)^{-1})]$. Composed with ι it is the identity on $\mathfrak{su}(2)$, charge conjugation $\eta \mapsto -\eta^T$ (an outer automorphism) on $\mathfrak{su}(3)$, and $+5$ on y . Complex conjugation is inner on $SU(2)$ and outer on $SU(3)$, which is why the sign moves to color. One of the two signs is unavoidable: each of the 24 root-datum isomorphisms of Sec. IIA, composed with ι_* , acts as $(\epsilon_2 | \epsilon_3 w | -5\epsilon_3)$ on $(\mathfrak{su}_2 | \mathfrak{su}_3 | y)$, with $w \in W(SU(3))$, six in each class. None is the identity on color with a positive hypercharge factor, and none has a factor other than ± 5 . On the labels of Sec. IIC the two dictionaries act as

$$\begin{aligned} \Psi \circ \iota : (a_1, a_2, n, y) &\mapsto (a_1, a_2, n, -5y), \\ \Psi_C \circ \iota : (a_1, a_2, n, y) &\mapsto (a_2, a_1, n, +5y); \end{aligned} \quad (13)$$

for example $Q_L = (1, 0, 1, 1)$ goes to $(1, 0, 1, -5)$ and to $(0, 1, 1, 5)$.

These dictionaries will be used to define the mirror sector in Sec. III. We note here what they do to the matter content. If the mirror hypercharge is normalized as in the ordinary sector, $Y'(Q'_L) = \frac{1}{6}$, which is the only normalization that gives a G_{SM} content, then Ψ gives the mirror sector the same representation labels R as the ordinary one, while Ψ_C gives the conjugate labels $\bar{R}$. Parity

also acts on the left-handed content as $R \mapsto \bar{R}$. The C-twist and parity therefore cancel: combined, they give the identical labels, not the mirror labels. The helicity structure of the mirror sector must come from elsewhere, and we return to it in Sec. III.

E. Generalizations

The construction extends to $S(U(m) \times U(n))$, with the same lattices (1) on $\mathbb{Z}^{m+n}$. Let $d = \gcd(m, n)$. The center of $S(GL_m \times GL_n)$ has component group $\mathbb{Z}_d$, $X^*/\mathbb{Z}\Phi \cong \mathbb{Z} \oplus \mathbb{Z}_d$, while its fundamental group is $\mathbb{Z}$ for all m, n . Its Langlands dual is $(GL_m \times GL_n)/\mathbb{G}_m$, which has a connected center and $\pi_1 \cong \mathbb{Z} \oplus \mathbb{Z}_d$. For $d > 1$ the group is therefore not self-dual. The coprimality of m and n is necessary, but it is not sufficient:

$$\begin{aligned} S(U(m) \times U(n)) \text{ self-dual} \\ \iff \gcd(m, n) = 1, \quad n \equiv \pm 1 \pmod{m}, \\ m \equiv \pm 1 \pmod{n} \\ \iff \min(m, n) = 1 \text{ or } |m - n| = 1. \end{aligned} \quad (14)$$

The proof, a comparison of the ways the two blocks are glued in X^* and in X_* , is given in App. D. For example $(2, 5)$ is coprime but not self-dual. An isomorphism with $F\alpha = \alpha^\vee$ for all roots exists only for $\min(m, n) = 1$ and for $\{m, n\} = \{2, 3\}$: among $m, n \geq 2$, the Standard Model case is the only one. In each self-dual case the dual quotient is isomorphic to the subgroup, $(U(m) \times U(n))/U(1) \cong S(U(m) \times U(n))$, through explicit maps. For $n = m + 1$ one is

$$[(A, B)] \mapsto \left((A^T)^{-1} \frac{\det B}{\det A}, B \frac{\det A}{\det B} \right), \quad (15)$$

with inverse $(A, B) \mapsto [(A^T)^{-1}, B]$; on unitary matrices $(A^T)^{-1} = \bar{A}$. For all m, n , $S(U(m) \times U(n)) \cong (SU(m) \times SU(n) \times U(1))/\mathbb{Z}_{\text{lcm}(m, n)}$, which reduces to Eq. (6) for $(m, n) = (2, 3)$.

With k blocks, $N = \sum_i n_i$ and $d = \gcd(n_1, \dots, n_k)$, the center of $S(GL_{n_1} \times \dots \times GL_{n_k})$ has $X^*/\mathbb{Z}\Phi \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}_d$, the fundamental group is $\mathbb{Z}^{k-1}$ without torsion, and the Langlands dual is $(\prod_i GL_{n_i})/\mathbb{G}_m$; so $d = 1$ is again necessary. The criterion is

$$\begin{aligned} S(U(n_1) \times \dots \times U(n_k)) \text{ self-dual} \\ \iff n_j \equiv \pm 1 \pmod{\gcd\{n_i : i \neq j\}} \text{ for every } j, \end{aligned} \quad (16)$$

with $\gcd \emptyset = 0$; for $k = 2$ it reduces to Eq. (14). It is proved in App. D. The condition $d = 1$ is sufficient exactly when each n_j is ± 1 modulo $m_j = \gcd\{n_i : i \neq j\}$. This is automatic if every m_j lies in $\{1, 2, 3, 4, 6\}$, the moduli m with $(\mathbb{Z}/m)^\times = \{\pm 1\}$, for example if $k \geq 3$ and the n_i are pairwise coprime; it also holds whenever some $n_i = 1$. A scan of all 494 tuples with $k \leq 4$ and $n_i \leq 8$, each decided by an explicit isomorphism or by an obstruction, agrees with Eq. (16); 28 of the 407 tuples with $d = 1$ are not self-dual, e.g. $(2, 5, 5)$ and $(3, 7, 7)$.

The criterion also covers the three-block groups $S(U(2) \times U(3) \times U(k)) \cong G_{\text{SM}} \times U(k)$, $k = 2, 6$, which are self-dual and underlie the four-dimensional mirror models of the companion paper [30].

In summary, G_{SM} is Langlands self-dual through 24 explicit isomorphisms, it is the only self-dual one among the quotients of $SU(3) \times SU(2) \times U(1)$ by subgroups of $\mathbb{Z}_6$, and the identification of $G_{\text{SM}}^\vee$ with G_{SM} necessarily carries a twist, charge conjugation on color or a factor -5 on the hypercharge. Among the groups $S(U(n_1) \times \dots \times U(n_k))$ the property is not generic, and among two-block groups with both blocks larger than 1 the Standard Model case is the only one in which every root goes to its own coroot.

III. MIRROR DUALITY FROM ELECTRIC–MAGNETIC DUALITY

This section asks how S-duality, which relates the theory with gauge group G and coupling $\tau = \theta/2\pi + 4\pi i/g^2$ to the theory with group $G^\vee$ and coupling $-1/\tau$ [13, 15–17, 19], can map the ordinary sector of mirror matter onto the mirror sector, and what such a map implies for the couplings and the matter of the two sectors. Since S alone only relates two descriptions of one theory (Sec. I), we work on the product theory $T[G_{\text{SM}}, \tau] \otimes T[G'_{\text{SM}}, \tau']$ of an ordinary and a mirror sector, which share spacetime and nothing else, and combine S with the exchange of the two sectors. Throughout, S is a formal map on couplings, charge lattices and line operators; when it can hold for G_{SM} is the subject of App. B, and our conventions are those of App. A.

Three notions recur. (i) The *mirror maps* $\mathcal{M}$ and $\mathcal{M}_P$ of Eq. (18): both dualize each sector and exchange the two, and $\mathcal{M}_P$ in addition reverses the orientation of spacetime. (ii) The *frames*: since S changes which charges are called electric, we compare the original frame, in which each sector is described by its own electric variables, with the frames D and D' obtained by applying S or S^{-1} to the mirror sector alone; in D the magnetic charges of the mirror sector become electric. (iii) The *lifts*: maps that act in the same way on the couplings can differ on the charges by charge conjugation C of one sector; the lifts of $\mathcal{M}$ and $\mathcal{M}_P$ by $1 \otimes C$ are written $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$. A map can act on chiral matter only if it squares to 1 modulo Weyl transformations, and of the four only $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ do (Sec. III E).

A. Definitions and group structure

In our conventions $F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\mathbf{E} \cdot \mathbf{B}$ is odd under parity and $F_{\mu\nu}F^{\mu\nu}$ is even. Parity therefore acts on the coupling and on the charges $q = (n_m, n_e)$ as

$$P: \tau \mapsto -\bar{\tau}, \quad (n_m, n_e) \mapsto (-n_m, n_e), \quad (17)$$

for either sign of ϵ^{0123} or of the θ -term, in agreement with the action of orientation reversal in Ref. [19]. As a matrix, $P = \text{diag}(-1, 1) \in GL(2, \mathbb{Z})$; matrices act on $(\tau, 1)$, antiholomorphically when their determinant is -1 , and on charges as $q \mapsto q\gamma^{-1}$. We define two maps on the product theory,

$$\mathcal{M} = \text{swap} \circ (S \otimes S), \quad \mathcal{M}_P = \text{swap} \circ P \circ (S \otimes S), \quad (18)$$

where P acts on both sectors (they share spacetime) and maps compose from right to left. They act on the couplings as

$$\begin{aligned} \mathcal{M}: (\tau, \tau') &\mapsto \left(-\frac{1}{\tau}, -\frac{1}{\tau'}\right), \\ \mathcal{M}_P: (\tau, \tau') &\mapsto \left(\frac{1}{\bar{\tau}}, \frac{1}{\bar{\tau}'}\right), \end{aligned} \quad (19)$$

so that each fixes the couplings on its fixed locus (with S a formal map; S-duality is expected to be exact only in the $\mathcal{N} = 4$ theory, App. B),

$$\text{Fix}(\mathcal{M}) = \{\tau' = -1/\tau\}, \quad \text{Fix}(\mathcal{M}_P) = \{\tau' = 1/\bar{\tau}\}. \quad (20)$$

On both loci $4\pi/g'^2 = \text{Im } \tau/|\tau|^2$ and $\theta'/2\pi = \mp \text{Re } \tau/|\tau|^2$, with the upper sign for $\mathcal{M}$ and the lower one for $\mathcal{M}_P$. Since $-1/\tau - 1/\bar{\tau} = -2\text{Re } \tau/|\tau|^2$, the two loci meet exactly at $\theta = 0$, where $\theta' = 0$ and both read

$$g g' = 4\pi. \quad (21)$$

For a single factor, modulo $T: \tau' \mapsto \tau' + 1$ acting on τ' alone, they also meet where $2\text{Re } \tau/|\tau|^2 \in \mathbb{Z}$, for example at $\tau = e^{i\pi/3}$. For a rescaled, Fricke-type duality $S_N: \tau \mapsto -1/(N\tau)$ the loci become $\tau' = -1/(N\tau)$ and $\tau' = 1/(N\bar{\tau})$, with $g g' = 4\pi\sqrt{N}$ at $\theta = 0$; this form will appear for the hypercharge in Sec. III B. There the factors $N_Y = 9$ and $N_1 = 25$ of Eq. (28) come from the normalization of the hypercharge generator and from the $\mathbb{Z}_6$ glue, not from roots of different lengths as for non-simply-laced groups [31–33]; all simple factors of G_{SM} are simply laced. The locus $\tau' = -1/\tau$ is the operational content of the statement “ordinary electric $\leftrightarrow$ mirror magnetic”; for line operators it is made precise in Sec. IV.

In one sector, with S and T as in App. A and $C = -1$,

$$\begin{aligned} S^2 &= C, & P^2 &= 1, & PSP^{-1} &= S^{-1} = CS, \\ PTP^{-1} &= T^{-1}, & (PS)^2 &= 1, \end{aligned} \quad (22)$$

and $\langle S, P \rangle$ is the dihedral group D_4 of order 8, with center $\{1, C\}$; it is the stabilizer of $\tau = i$ in $GL(2, \mathbb{Z})$ and acts on τ through $\mathbb{Z}_2 \times \mathbb{Z}_2$. On the product, the maps (18) lie in $(SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})) \rtimes \langle \text{swap}, P \otimes P \rangle$ and satisfy

$$\begin{aligned} \mathcal{M}^2 &= C \otimes C, & \mathcal{M}_P^2 &= 1, \\ \mathcal{M}\mathcal{M}_P &= P \otimes P, & \mathcal{M}_P\mathcal{M}\mathcal{M}_P^{-1} &= \mathcal{M}^{-1}. \end{aligned} \quad (23)$$

Thus $\mathcal{M}$ generates a $\mathbb{Z}_4$, $\mathcal{M}_P$ is an involution, and $\langle \mathcal{M}, \mathcal{M}_P \rangle = \langle \mathcal{M}, P \otimes P \rangle \cong D_4$, with center $\{1, C \otimes C\}$.

Its image on (τ, τ') is $\{1, \mathcal{M}, \mathcal{M}_P, P \otimes P\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The element $\mathcal{M}$ preserves the Dirac pairing of the product theory, and $\mathcal{M}_P$ reverses it. The other ordering, $\text{swap} \circ (S \otimes S) \circ P = \mathcal{M}_P(C \otimes C)$, is also an involution. The lift matters: $\tilde{\mathcal{M}} = \text{swap} \circ (S \otimes S^{-1}) = \mathcal{M}(1 \otimes C)$ acts on the couplings as $\mathcal{M}$ does, but has order 2.

The two maps take a familiar form in the S-dual frame of the mirror sector, the frame $D = 1 \otimes S$ in which the mirror sector is described by its dual variables:

$$\begin{aligned} DMD^{-1} &= \text{swap} \circ (C \otimes 1), \\ D\mathcal{M}_PD^{-1} &= \text{swap} \circ (P \otimes P), \end{aligned} \quad (24)$$

and D maps $\text{Fix}(\mathcal{M})$ to $\{\tau' = \tau\}$ and $\text{Fix}(\mathcal{M}_P)$ to $\{\tau' = -\tau\}$. If per-sector S-duality is an exact equivalence, then in the frame D , $\mathcal{M}_P$ is the standard mirror parity of Refs. [4, 5], $\text{swap} \circ (P \otimes P)$, the exchange of the sectors combined with spacetime parity, written in an electric frame for the ordinary sector and a magnetic frame for the mirror sector; it is the natural choice, since it reverses the orientation of both spacetime and the internal space. The map $\mathcal{M}$ is then a charge-conjugated exchange. This reading is an interpretation of Eq. (24), not a derived statement; its consequences, and the case in which S is not an exact equivalence, are discussed in Sec. III H.

On the charge lattice $X^* \oplus X_*$ of G_{SM} , S followed by the transport back to G_{SM} with a root-datum isomorphism $\Phi : X^* \rightarrow X_*$ (Sec. II A) is the charge map of Eq. (A7),

$$S_\Phi(e, m) = (-\Phi^{-T}m, \Phi e), \quad (25)$$

and we use this form throughout. Transporting instead with Φ^{-1} on the electric and Φ^T on the magnetic charges gives the map (25) at the transposed isomorphism Φ^T , again one of the 24; the two forms agree for symmetric Φ , agree modulo W for every Φ , and run over the same 24 maps. For all 24 isomorphisms, S_Φ is integral and unimodular, preserves the Dirac pairing $e \cdot m' - m \cdot e'$, and normalizes W ; $S_\Phi^2 = C$ if and only if Φ is symmetric, and S_Φ has order 4 modulo W for all 24. Φ is symmetric for 16 of the 24, among them F of Eq. (2) and the four W -invariant forms; for these S_Φ has order 4, $PS_\Phi P^{-1} = S_\Phi^{-1}$ with $P(e, m) = (e, -m)$, and $\langle S_\Phi, P \rangle \cong D_4$, as in Eq. (22). For $\Phi = F$, $(\alpha, 0) \mapsto (0, \alpha^\vee)$ and $(0, \alpha^\vee) \mapsto (-\alpha, 0)$ for every root α . The lattice versions $\mathcal{M} = \text{swap}(S_\Phi \otimes S_\Phi)$ and $\mathcal{M}_P = \text{swap}(P \otimes P)(S_\Phi \otimes S_\Phi)$ then satisfy $\mathcal{M}^2 = C \otimes C$, $\mathcal{M}_P^2 = 1$ and $\mathcal{M}\mathcal{M}_P = P \otimes P$, generate a group of order 8, and take the frame forms (24) with $D = 1 \otimes S_\Phi$. For the 8 non-symmetric Φ , S_Φ has order 12 and $PS_\Phi P^{-1} = -S_\Phi \neq S_\Phi^{-1}$, and these relations fail; modulo $W \times W'$, however, $\mathcal{M}^2 \equiv C \otimes C$ and $\mathcal{M}_P^2 \equiv 1$ hold for every Φ (Sec. III E). The element $C = -1$ is a root-datum automorphism of G_{SM} that is not in W , i.e. an outer automorphism.

B. Couplings at $\theta = 0$

At $\theta = 0$ we write the Cartan part of the gauge kinetic term as $\mathcal{L} = -\frac{1}{4}h(\mathcal{F}, \mathcal{F})$, with the coupling absorbed

in the field strength, so that h is a positive invariant form on the Cartan algebra $\mathfrak{t}$, magnetic fluxes lie in $2\pi X_*$ and electric charges in X^* . Matching the electric and magnetic Coulomb energies shows that S inverts this form on the dual space,

$$h^\vee = \frac{1}{4\pi^2} h^{-1} \quad \text{on } \mathfrak{t}^* = \mathfrak{t}^\vee. \quad (26)$$

For a torus of any rank this is the unique solution, it is an involution, and it is the Siegel form of $\tau = 2\pi i h \mapsto -\tau^{-1}$. It gives $ee' = 2\pi$ for a $U(1)$ with integer charges in the normalization of Eq. (A3), and, with the roots of G identified with the coroots of $G^\vee$, $gg' = 4\pi$ for $SU(2)$ and $SU(3)$, in agreement with $\tau \mapsto -1/\tau$.

For G_{SM} the dual form lives on $\mathfrak{t}^*$ of $G_{\text{SM}}^\vee$ and is transported to the Cartan algebra of G_{SM}' by a root-datum isomorphism Φ , $h' = \Phi^{-T}h^{-1}\Phi^{-1}/4\pi^2$. We normalize the hypercharge in both sectors as in App. A, $Y(Q_L) = Y'(Q'_L) = \frac{1}{6}$, so that the factor -5 of Eq. (10) goes into the couplings. For each of the 24 isomorphisms, including the dictionaries Ψ and Ψ_C of Sec. II D, h' is again of Standard Model type, with

$$\begin{aligned} g_3 g'_3 &= g_2 g'_2 = 4\pi, & g_Y g'_Y &= 12\pi, \\ g_1 g'_1 &= 20\pi & (g_1 &= \sqrt{5/3} g_Y). \end{aligned} \quad (27)$$

These relations are $\text{Fix}(\mathcal{M}) = \text{Fix}(\mathcal{M}_P)$ at $\theta = 0$. The hypercharge factor $20\pi = 4\pi \cdot 5$ is the $|-5|$ of $\mathbf{M} = FK$ on y [Eq. (11)]. The single-sector self-dual point is unique: $g_3 = g_2 = \sqrt{4\pi}$ and $g_1 = \sqrt{20\pi}$ ($g_Y = \sqrt{12\pi}$), i.e. $\alpha_3 = \alpha_2 = 1$, $\alpha_1 = 5$ and $\alpha_Y = 3$; there F is an isometry from $(X^*, h^\vee)$ to (X_*, h) . Away from this point the relations pair a weakly coupled ordinary sector with a mirror sector that is strongly coupled in its electric frame, the starting point of the phenomenology of Sec. V.

The hypercharge relation depends on the normalization of the generator. With generators $c\hat{Y}$ and $c'\hat{Y}'$ and couplings $g_c, g'_{c'}$, the transport gives $g_c g'_{c'} = 12\pi/(cc')$, i.e. 12π for (g_Y, g'_Y) and 20π for (g_1, g'_1) . With $\tau_k = \theta_k/2\pi + 4\pi i/g_k^2$ for each factor, $\mathcal{M}$ at $\theta = 0$ is the Fricke form

$$\tau'_k = -\frac{1}{N_k \tau_k}, \quad N_3 = N_2 = 1, \quad N_Y = 9, \quad N_1 = 25, \quad (28)$$

equivalently, the lattice-normalized coupling $\hat{\tau}_Y$ of Eq. (A5) transforms by the plain S . Without the $\mathbb{Z}_6$ quotient, i.e. for the cover $SU(3) \times SU(2) \times U(1)$ with a separate duality of the $U(1)$ factor on integer y charges, one would instead find $g_Y g'_Y = 72\pi$; the ratio $6 = [X_*(G_{\text{SM}}) : \text{coroots} \oplus \mathbb{Z}y]$ is the index of the glue. None of these relations depends on the choice among the 24 isomorphisms or on the sign of S .

C. An abelian model with kinetic mixing

Kinetic mixing [34] between the ordinary and mirror photons is a phenomenologically relevant portal between the two sectors [5]. To see how $\mathcal{M}$ and $\mathcal{M}_P$

constrain it, we first take two $U(1)$ gauge fields, one per sector, with integer charges, couplings e, e' , θ -angles θ, θ' , kinetic mixing ϵ and a mixed θ -term θ_{mix} , normalized as in App. A. In charge-normalized fields F^I ($I = 1$ ordinary, $I = 2$ mirror) the Lagrangian is $\mathcal{L} = -\frac{1}{4}K_{IJ}F_{\mu\nu}^IF_{\mu\nu}^J + \frac{\Theta_{IJ}}{16\pi^2}F_{\mu\nu}^I\tilde{F}^{J\mu\nu}$, and the equations of motion and the dyon energies depend on the couplings only through

$$\tau = \frac{\Theta}{2\pi} + 2\pi i K = \begin{pmatrix} \frac{\theta}{2\pi} + \frac{2\pi i}{e^2} & \frac{\theta_{\text{mix}}}{2\pi} + \frac{2\pi i \epsilon}{ee'} \\ \frac{\theta_{\text{mix}}}{2\pi} + \frac{2\pi i \epsilon}{ee'} & \frac{\theta'}{2\pi} + \frac{2\pi i}{e'^2} \end{pmatrix}. \quad (29)$$

The duality group is $Sp(4, \mathbb{Z})$, acting as $\tau \mapsto (A\tau + B)(C\tau + D)^{-1}$, with $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, and on the charges $q = (m, n) \in \mathbb{Z}^2 \oplus \mathbb{Z}^2$ as $q \mapsto q\gamma^{-1}$, extended by parity to $Sp^\pm(4, \mathbb{Z})$, whose anti-symplectic elements act anti-holomorphically. The symplectic action on a matrix of abelian couplings is that of Refs. [35, 36]; here the off-diagonal entry of τ carries ϵ and θ_{mix} . Parity sends $\Theta \rightarrow -\Theta$. The group $(SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})) \rtimes \langle \text{swap} \rangle$ of Sec. III A embeds as the stabilizer of the sector decomposition, and the four elements we need act as

$$\begin{aligned} \mathcal{M} : \tau &\mapsto -\sigma_x \tau^{-1} \sigma_x, & \mathcal{M}_P : \tau &\mapsto \sigma_x \bar{\tau}^{-1} \sigma_x, \\ \tilde{\mathcal{M}} : \tau &\mapsto -\tau / \det \tau, & \tilde{\mathcal{M}}_P : \tau &\mapsto \bar{\tau} / \det \bar{\tau}, \end{aligned} \quad (30)$$

with the lifts $\tilde{\mathcal{M}} = \mathcal{M}(1 \otimes C) = \text{swap}(S \otimes S^{-1})$ and $\tilde{\mathcal{M}}_P = \mathcal{M}_P(1 \otimes C)$. Their orders in $Sp^\pm(4, \mathbb{Z})$ are 4, 2, 2 and 4 for $\mathcal{M}$, $\mathcal{M}_P$, $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$, with $\mathcal{M}^2 = \tilde{\mathcal{M}}_P^2 = C \otimes C$; on τ all four act as involutions, since $C \otimes C$ acts trivially. Because $C \otimes 1$ sends $\tau_{12} \mapsto -\tau_{12}$, the lifts act differently from $\mathcal{M}$ and $\mathcal{M}_P$ as soon as $\tau_{12} \neq 0$.

The fixed loci are exact constraints on the four real couplings and the two mixing parameters. The element $\mathcal{M}$ fixes $\{\tau_{12} = 0, \tau_{22} = -1/\tau_{11}\}$: it forces $\epsilon = \theta_{\text{mix}} = 0$ for all couplings, and the rest is the locus (20), with $ee' = 2\pi$ at $\theta = 0$. The element $\mathcal{M}_P$ fixes $\{\text{Re}(\tau_{11}\bar{\tau}_{12}) = 0, \tau_{11}\bar{\tau}_{22} + |\tau_{12}|^2 = 1\}$, which in couplings reads

$$\begin{aligned} \epsilon &= -\frac{\theta \theta_{\text{mix}} e^3 e'}{16\pi^4}, & \theta e^2 &= \theta' e'^2, \\ \frac{\theta \theta' + \theta_{\text{mix}}^2}{4\pi^2} + \frac{4\pi^2(1 + \epsilon^2)}{(ee')^2} &= 1. \end{aligned} \quad (31)$$

Kinetic mixing invariant under $\mathcal{M}_P$ thus requires $\theta \theta_{\text{mix}} \neq 0$. At $\theta = \theta' = 0$, $\epsilon = 0$, $|\theta_{\text{mix}}| < 2\pi$ and $ee' = 2\pi/\sqrt{1 - \theta_{\text{mix}}^2/4\pi^2}$. For the lifts, $\text{Fix}(\tilde{\mathcal{M}}) = \{\det \tau = -1\}$, and $\text{Fix}(\tilde{\mathcal{M}}_P) = \{\text{Im}(\bar{\tau}_{11}\tau_{12}) = 0, \bar{\tau}_{11}\tau_{22} - |\tau_{12}|^2 = 1\}$, i.e.

$$\begin{aligned} \theta_{\text{mix}} &= \frac{\theta e}{e'}, & \theta e^2 &= \theta' e'^2, \\ \frac{\theta \theta' - \theta_{\text{mix}}^2}{4\pi^2} + \frac{4\pi^2(1 - \epsilon^2)}{(ee')^2} &= 1. \end{aligned} \quad (32)$$

At $\theta = \theta' = 0$ the locus of $\tilde{\mathcal{M}}_P$ gives $\theta_{\text{mix}} = 0$ and $ee' = 2\pi\sqrt{1 - \epsilon^2}$ with ϵ free, while that of $\mathcal{M}$ requires

$\epsilon \theta_{\text{mix}} = 0$, with $ee' = 2\pi\sqrt{1 - \epsilon^2}$ ($\theta_{\text{mix}} = 0$) or $ee' = 2\pi/\sqrt{1 - \theta_{\text{mix}}^2/4\pi^2}$ ($\epsilon = 0$). At $\theta = 0$, kinetic mixing is therefore allowed by $\tilde{\mathcal{M}}_P$ (order 4) and by $\tilde{\mathcal{M}}$ (order 2), and forbidden by $\mathcal{M}$ for all couplings and by $\mathcal{M}_P$.

The conjugacy (24) explains these loci. With $D = 1 \otimes S$,

$$\begin{aligned} D\mathcal{M}D^{-1} &= \text{swap}(C \otimes 1), \\ D\mathcal{M}_PD^{-1} &= \text{swap}(P \otimes P), \\ D\tilde{\mathcal{M}}D^{-1} &= \text{swap}(C \otimes C), \\ D\tilde{\mathcal{M}}_PD^{-1} &= \text{swap}(P \otimes P)(1 \otimes C), \end{aligned} \quad (33)$$

and $\text{swap}(C \otimes C)$ acts on τ as the plain swap. The locus of the classical mirror parity swap $(P \otimes P)$ is $e' = e$, $\theta' = -\theta$, $\theta_{\text{mix}} = 0$ with ϵ free, and that of $\text{swap}(P \otimes P)(1 \otimes C)$ is $e' = e$, $\theta' = -\theta$, $\epsilon = 0$ with θ_{mix} free. Theories invariant under $\mathcal{M}_P$ are thus the S-dual frames, in the mirror sector, of mirror-parity-invariant theories with arbitrary ϵ . That an S transformation of one $U(1)$ factor turns a mixed θ -term (“magnetic mixing”) into kinetic mixing, with the same effective charges, was shown for a visible and a hidden $U(1)$ in Ref. [37]; kinetic mixing of two abelian theories with both electric and magnetic charges, with a manifest diagonal $SL(2, \mathbb{Z})$, is studied in Ref. [38], and the millimagnetic charges that kinetic mixing generically produces are bounded in Ref. [39]. Here this exchange is combined with the exchange of the sectors.

D. The $\mathbb{Z}_6$ -glued model

We now replace each $U(1)$ by the Cartan torus of G_{SM} , with the electric lattice Λ_e of Eq. (7) and Weyl-invariant couplings. In one sector the coupling on X_* is

$$\tau = \tau_3 \text{tr}_3 + \tau_2 \text{tr}_2 + \frac{\hat{\tau}_Y}{6} \ell_Y \otimes \ell_Y, \quad (34)$$

with $\ell_Y(\lambda) = \tilde{Y}$ the magnetic hypercharge coordinate of Sec. II C, $\text{tr}_{3,2}$ the trace forms of the two simple factors, $\tau_k = \theta_k/2\pi + 4\pi i/g_k^2$, and $\hat{\tau}_Y = 3\theta_Y/2\pi + 12\pi i/g_Y^2 = 5\theta_1/2\pi + 20\pi i/g_1^2$. The Standard Model weights span exactly X^* , which has index 6 in the lattice of the cover. The duality group that commutes with W and acts on $(\tau_3, \tau_2, \hat{\tau}_Y)$ is

$$\Gamma_{\text{glue}} = \{(\gamma_3, \gamma_2, \gamma_Y) \in SL(2, \mathbb{Z})^3 : \gamma_3 \equiv U\gamma_Y U \pmod{3}, \gamma_2 \equiv \gamma_Y \pmod{2}\}, \quad (35)$$

with $U = \text{diag}(1, -1)$. It has index 144 in $SL(2, \mathbb{Z})^3$ and equals $\delta(SL(2, \mathbb{Z})) \cdot (\Gamma(3) \times \Gamma(2) \times 1)$ with $\delta(\gamma) = (U\gamma U, \gamma, \gamma)$; in particular S on a single factor is not in Γ_{glue} (App. B). Every root-datum isomorphism Φ gives an element S_Φ acting as $(\tau_3, \tau_2, \hat{\tau}_Y) \mapsto (-1/\tau_3, -1/\tau_2, -1/\hat{\tau}_Y)$, which is Eq. (28). The four W -equivariant ones are $S_\Phi = (s_3 S, s_2 S, s_Y S)$ with $s_3 s_Y = -1$, the four classes of Sec. II D, so $(S, S, S) \notin \Gamma_{\text{glue}}$: a

C-twist on $SU(3)$ or on the hypercharge is forced by the glue. This is the lattice form of the factor -5 of Eq. (11).

The θ -angles inherit the glue. On spin four-manifolds the periods of the nonabelian $\mathbb{Z}_6$ -glued theory, obtained from its instanton numbers, are

$$(\Delta\theta_3, \Delta\theta_2, \Delta\theta_Y) = 2\pi(b_3, b_2, \frac{b_Y}{3}), \quad (36)$$

$$b_Y \equiv -b_3 \pmod{3}, \quad b_Y \equiv b_2 \pmod{2},$$

or $\Delta\theta_1 = 2\pi b_Y/5$ in grand-unified normalization. They form the lattice Λ_θ generated by $2\pi(1, 1, -\frac{1}{5})$ in $(\theta_3, \theta_2, \theta_1)$ and the pure periods $(6\pi, 0, 0)$, $(0, 4\pi, 0)$ and $(0, 0, \frac{12\pi}{5})$ (4π for θ_Y), and $\Lambda_\theta/\Lambda_{\text{axes}} \cong \mathbb{Z}_6$. The $\mathbb{Z}_3$ part ties θ_3 to θ_Y with the sign of the C-twist and the $\mathbb{Z}_2$ part ties θ_2 to θ_Y ; in particular, $\theta_3 \rightarrow \theta_3 + 2\pi$ alone is not a symmetry. This lattice coincides with the Witten-effect lattice [40] of the Cartan theory, so that instantons add no constraint. For comparison, the cover has no correlation (a θ_Y period of $\pi/9$), and the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ quotients impose one of the two congruences each. The dependence of the θ periods and of axion-gauge couplings on the global form of the Standard Model group is discussed in Refs. [41, 42], and discrete θ angles of quotient groups in Ref. [43]. The Standard Model requires spin manifolds, since $(-1)^F \notin G_{\text{SM}}$.

The periods (36) treat θ_3 , θ_2 and θ_Y as couplings of the gauge theory in a fixed basis of the fermion fields: anomalous chiral rotations of the fermions, which shift the θ -angles, are not used. With chiral matter the individual angles are then coordinates of a redundant parametrization, and we now identify the physical combinations. For a left-handed Weyl fermion of charge Q , and for a rotation of left-handed multiplets with charges X ,

$$\partial_\mu j_L^\mu = +\frac{Q^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (37)$$

$$\psi \rightarrow e^{i\alpha X} \psi : \quad \Delta\theta_G = -2\alpha \sum_{\text{LH}} X T_G(R),$$

with $T_Y(R) = Y^2 \dim R$ for the hypercharge, the θ -terms of App. A, and the opposite sign for right-handed multiplets; the anomaly agrees with Eq. (19.59) of Ref. [44]. [45] A 2π rotation of a single multiplet is the identity on the fields, and the shifts of such rotations of the five multiplets of one generation generate exactly Λ_θ , so the periods are consistent with the rotations. Together with the flavor rotations and the rephasing of the Higgs field, the rotations act on the three angles and on the phases $\varphi_f = \arg \det Y_f$ of the Yukawa matrices in $\mathcal{L} \supset -Y_u Q u^c H - Y_d Q d^c H^\dagger - Y_e L e^c H^\dagger + \text{h.c.}$, written with left-handed Weyl fields. For three generations, one Higgs doublet, generic Yukawa matrices and no right-handed neutrinos they leave exactly two invariants per sector,

$$\begin{aligned} \bar{\theta} &= \theta_3 + \varphi_u + \varphi_d, \\ I_{\text{EW}} &= \theta_2 + \theta_Y - \frac{2}{3}\theta_3 + 2(\varphi_u + \varphi_e), \\ \bar{\theta}' &= \theta'_3 - \varphi'_u - \varphi'_d, \\ I'_{\text{EW}} &= \theta'_2 + \theta'_Y - \frac{2}{3}\theta'_3 - 2(\varphi'_u + \varphi'_e), \end{aligned} \quad (38)$$

with $\bar{\theta}$ defined modulo 2π and I_{EW} modulo 4π , and θ_{mix} , which no rotation shifts since no field carries both Y and Y' . Here the mirror matter is the right-handed parity image of the ordinary one, with the mirror Yukawa matrices multiplying products of right-handed fields. Equivalently, $\bar{\theta} = \theta_3 + \arg \det(M_u M_d)$ and $\bar{\theta}' = \theta'_3 - \arg \det(M'_u M'_d)$ for quark mass terms $-(M_q)_{ij} q_i q_j^c + \text{h.c.}$ and their mirror images. On $\mathbb{R}^4$ the rotations that preserve the Yukawa couplings, B and L , shift the angles only along $(0, 1, -1)$ in $(\theta_3, \theta_2, \theta_Y)$ (App. B), so θ_2 can be moved into θ_Y , which drops out when all hypercharge bundles are trivial: only $\bar{\theta}$ is observable there, while I_{EW} acts as a coupling only on topologically nontrivial backgrounds, such as 't Hooft lines or hypercharge flux on spin manifolds. The rotations shift $(\theta_3, \theta_2, \theta_Y)$ with rank 3, so $\theta_3 = \theta_2 = \theta_Y = 0$ can be reached in each sector independently, with $\bar{\theta}$ and I_{EW} carried by the Yukawa phases. We call these three angles per sector the removable angles; θ_{mix} is not removable and remains a coupling. The relations at $\theta = 0$ of this paper, such as Eq. (27), refer to the slice on which the removable angles vanish, while the loci with $\theta \neq 0$ below, Eqs. (40)–(42), and the Witten effect of Eq. (52) hold in a fixed fermion basis. What they say about $\bar{\theta}$ and $\bar{\theta}'$ is discussed in Sec. III H.

With two sectors, the only $W \times W'$ -invariant couplings between them are the hypercharge kinetic mixing and θ_{mix} of Eqs. (A4) and (A6). They enter the coupling matrix as $\tau_{12} = \frac{1}{6} \hat{\tau}_{\text{mix}} \ell_Y \otimes \ell'_Y$, and the hypercharge block is

$$\hat{\tau} = \begin{pmatrix} \hat{\tau}_Y & \hat{\tau}_{\text{mix}} \\ \hat{\tau}_{\text{mix}} & \hat{\tau}'_Y \end{pmatrix}, \quad \hat{\tau}_{\text{mix}} = \frac{3\theta_{\text{mix}}}{2\pi} + \frac{12\pi i \epsilon}{g_Y g'_Y}. \quad (39)$$

The duality group of the two sectors acts on $\hat{\tau}$ through $Sp(4, \mathbb{Z})$ elements congruent modulo 6 to a pair $\text{diag}(\gamma_Y, \gamma'_Y)$, each glued to its own sector as in Eq. (35). In particular θ_{mix} is periodic only under shifts in $4\pi\mathbb{Z}$ ($\frac{12\pi}{5}\mathbb{Z}$ in grand-unified normalization), compared with $\pi/9$ in the normalization of the cover, and the periods factorize as $\Lambda_\theta \times \Lambda'_\theta \times \{\Delta\theta_{\text{mix}} \in 4\pi\mathbb{Z}\}$, with no correlation between the sectors.

The elements $\mathcal{M} = \text{swap}(S_\Phi \otimes S_{\Phi'})$ and $\mathcal{M}_P = \text{swap}(P \otimes P)(S_\Phi \otimes S_{\Phi'})$ lie in this group and act on the simple factors as in Sec. III A. On the hypercharge block they act as the abelian $\mathcal{M}$ and $\mathcal{M}_P$ of Sec. III C when the two dictionaries are in the same class, $s_Y = s'_Y$, and as $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$ when one is of Ψ type (-5 on y) and the other of Ψ_C type (C on $SU(3)$, $+5$ on y). We use $\mathcal{M}, \mathcal{M}_P, \tilde{\mathcal{M}}, \tilde{\mathcal{M}}_P$ for the four resulting classes of elements on $G_{\text{SM}} \times G'_{\text{SM}}$. Their fixed loci are as follows. On the simple factors, $\mathcal{M}$ and $\tilde{\mathcal{M}}$ give $\tau'_k = -1/\tau_k$, so $g_k g'_k = 4\pi$ at $\theta_k = 0$, while $\mathcal{M}_P$ and $\tilde{\mathcal{M}}_P$ give

$$\theta'_k g_k'^2 = \theta_k g_k^2, \quad \frac{\theta_k \theta'_k}{4\pi^2} + \frac{16\pi^2}{(g_k g'_k)^2} = 1. \quad (40)$$

On the hypercharge block, $\mathcal{M}$ forces $\epsilon = \theta_{\text{mix}} = 0$ for all couplings and $\hat{\tau}'_Y = -1/\hat{\tau}_Y$, hence $g_Y g'_Y = 12\pi$ at $\theta_Y = 0$

(which forces $\theta'_Y = 0$). For $\mathcal{M}_P$,

$$\epsilon = -\frac{\theta_Y \theta_{\text{mix}} g_Y^3 g'_Y}{64\pi^4}, \quad \theta_Y g_Y^2 = \theta'_Y g_Y'^2, \quad (41)$$

$$\frac{9(\theta_Y \theta'_Y + \theta_{\text{mix}}^2)}{4\pi^2} + \frac{144\pi^2(1 + \epsilon^2)}{(g_Y g'_Y)^2} = 1,$$

so that at $\theta_Y = 0$, $\epsilon = 0$ and $g_Y g'_Y = 12\pi/\sqrt{1 - 9\theta_{\text{mix}}^2/4\pi^2}$. For $\tilde{\mathcal{M}}_P$,

$$\theta_{\text{mix}} = \frac{\theta_Y \epsilon g_Y}{g'_Y}, \quad \theta_Y g_Y^2 = \theta'_Y g_Y'^2, \quad (42)$$

$$\frac{9(\theta_Y \theta'_Y - \theta_{\text{mix}}^2)}{4\pi^2} + \frac{144\pi^2(1 - \epsilon^2)}{(g_Y g'_Y)^2} = 1,$$

so that at $\theta_Y = 0$, $\theta_{\text{mix}} = 0$ and $g_Y g'_Y = 12\pi\sqrt{1 - \epsilon^2}$ with ϵ free. Finally, $\tilde{\mathcal{M}}$ fixes $\det \hat{\tau} = -1$, which at $\theta_Y = \theta'_Y = 0$ requires $\epsilon \theta_{\text{mix}} = 0$, with $g_Y g'_Y = 12\pi\sqrt{1 - \epsilon^2}$ ($\theta_{\text{mix}} = 0$) or $g_Y g'_Y = 12\pi/\sqrt{1 - 9\theta_{\text{mix}}^2/4\pi^2}$ ($\epsilon = 0$). The glue does not change the shapes of these loci, which are those of Sec. III C with the effective lattice coupling $e^2 = g_Y^2/6$ and angle $3\theta_Y$. Kinetic mixing at $\theta_Y = \theta'_Y = \theta_{\text{mix}} = 0$ is allowed only by $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$, that is, exactly when the two dictionaries differ by the C-twist. Like the periods, all these loci treat the θ -angles as couplings in a fixed fermion basis.

The full duality group of the two-sector coupling matrix $\Omega = \text{diag}(\tau_3, \tau_2, \tau'_3, \tau'_2) \oplus \hat{\tau}$ is larger than the product of the sector groups. Taken as the integral (anti)symplectic normalizer of $W \times W'$, it is

$$N = (W \times W') \cdot \Gamma_2 \cdot \langle X_3, X_2, P \otimes P \rangle, \quad (43)$$

where Γ_2 is the group of integral elements described above and $N/((W \times W')\Gamma_2) \cong \mathbb{Z}_2^3$. The partial swaps X_3 and X_2 exchange only the $SU(3)$ or only the $SU(2)$ couplings of the two sectors, together with an integral transformation A_3 or A_2 of $\hat{\tau}$ that always generates hypercharge kinetic mixing; they are classical root-datum automorphisms of $G_{\text{SM}} \times G'_{\text{SM}}$, and $X_3 X_2 = \text{swap}$. All four mirror elements lie in the class of the swap: $\mathcal{M}$ and $\tilde{\mathcal{M}}$ in the coset $\text{swap} \cdot \Gamma_2$, and $\mathcal{M}_P$ and $\tilde{\mathcal{M}}_P$ in $\text{swap}(P \otimes P) \cdot \Gamma_2$. Modulo $W \times W'$ their orders are 4 ($\mathcal{M}$), 2 ($\tilde{\mathcal{M}}$), 2 ($\mathcal{M}_P$) and 4 ($\tilde{\mathcal{M}}_P$); the orders of individual elements in N depend on Φ and Φ' (Sec. III E). Matter removes the partial swaps. With Standard Model matter in both sectors, no element of the cosets of X_3 or X_2 , classical or not, maps the matter content to itself: the charge of Q_L would be mapped to one charged under both $SU(3)'$ and $SU(2)$ (or $SU(3)$ and $SU(2)'$), and X_3 would give L_L , H and e_R hypercharge in both sectors. Modulo $W \times W'$, the classical matter-preserving elements are $\langle \text{swap}, CP \rangle \cong \mathbb{Z}_2^2$, with $CP = (C \otimes C)(P \otimes P)$, if the second sector is an identical copy, and $\langle \text{swap}(P \otimes P), CP \rangle \cong \mathbb{Z}_2^2$ if its fermions have the opposite chirality. The sector exchange that survives mirror matter is thus the full swap combined with P , the coset of $\mathcal{M}_P$ and $\tilde{\mathcal{M}}_P$.

E. Chirality

Mirror matter pairs left-handed ordinary fermions with right-handed mirror fermions. A map between the sectors must reproduce this rule to act on the chiral Standard Model, and a natural question is whether the C-twist forced in the dictionary (Sec. II D), combined with P , is its origin. We test this on all $24 \times 24 \times 2$ elements $X = \text{swap}(P \otimes P)^p(S_\Phi \otimes S_{\Phi'})$, $p \in \{0, 1\}$, of the glued two-sector lattice, with S_Φ as in Eq. (25), and with one chiral Standard Model generation in each sector. Each S_Φ lies in one of two twist classes, A (type Ψ , -5 on y , containing F) or B (type Ψ_C , C on $SU(3)$), twelve in each, and none is (S, S, S) modulo W . Class A consists of the $\Phi = Fw$ and class B of the $\Phi = -Fw$, $w \in W$, and S_{Fw} is S_F composed with the action of w on the charges, so that the S_Φ of one class coincide modulo W . The 1152 elements therefore reduce modulo $W \times W'$ to $2 \times 2 \times 2$, labeled by the two classes and by p , and we classify them modulo $W \times W'$. In the S-dual frames $D = 1 \otimes S_{\Phi'}$ and $D' = 1 \otimes S_{\Phi'}^{-1}$ of the mirror sector every DXD^{-1} is classical, and the result depends only on whether the two classes agree and on p ; it is summarized in Table II. An element is compatible with chiral matter if and only if $X^2 \equiv 1$ modulo $W \times W'$, i.e. for the same class with P ($\mathcal{M}_P$) or for mixed classes without P ($\tilde{\mathcal{M}}$), and it flips helicity if and only if $p = 1$. These statements, and Table II, hold for every Φ and Φ' , symmetric or not, and also with the transposed form of the transported S , which gives the same 1152 elements relabeled by $(\Phi, \Phi') \rightarrow (\Phi^T, \Phi'^T)$. The orders in the table are orders modulo $W \times W'$. In the full duality group the order of an element depends on Φ and Φ' : for $\Phi' = \Phi$ symmetric, $\mathcal{M}$ has order 4 and $\mathcal{M}_P$ order 2; for $\Phi' = \Phi$ non-symmetric, they have orders 12 and 6. On the $W \times W'$ -invariant couplings every one of the 1152 elements acts with order 2.

The answer to the question is therefore negative in its original form: the helicity flip comes from P alone. The mirror-parity form of $\mathcal{M}_P$ holds identically for the classes AA and BB, and also for an untwisted S (Sec. III C), so the per-sector twist drops out, while a relative twist combined with P ($\tilde{\mathcal{M}}_P$) is never compatible with chiral matter. $\mathcal{M}_P$ is matter compatible: in frame D it is $\text{swap}(P \otimes P)$, the exchange of left-handed ordinary and right-handed mirror fermions. Whether the second sector carries “mirror” ($\bar{R}$) or “copy” (R) labels changes between the frames D and D' , which differ by C on the mirror sector, and is a convention; the invariant is the orientation character, $\theta' = -\theta$ versus $\theta' = \theta$. For kinetic mixing, $\mathcal{M}$ and $\mathcal{M}_P$ force $\epsilon = 0$ on the slice $\theta_Y = \theta'_Y = 0$, while $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$ allow it. Chiral Standard Model matter together with $\epsilon \neq 0$ on this slice is therefore compatible only with $\tilde{\mathcal{M}}$, the helicity-preserving element with mismatched twist classes. At $\theta_Y \neq 0$, $\mathcal{M}_P$ also allows $\epsilon \neq 0$ [Eq. (41)]. These statements assume that the elements act as matter symmetries in an S-dual frame of the mirror sector, which is physical only if per-sector S-duality is an exact equivalence; the phenomenological consequences

TABLE II. The four classes of mirror elements on the $\mathbb{Z}_6$ -glued lattice of $G_{\text{SM}} \times G'_{\text{SM}}$ with one chiral Standard Model generation per sector. “Twist” says whether the dictionaries Φ, Φ' lie in the same class or not; “order” is the order modulo $W \times W'$ (the orders in the full duality group depend on Φ, Φ'); frames D and D' are the two S-dual frames of the mirror sector, with the content each form preserves (mirror: opposite chirality; copy: identical); the last column says whether hypercharge kinetic mixing $\epsilon \neq 0$ is allowed at $\theta_Y = \theta'_Y = 0$. The frame forms are labels on the charge lattice; on fermions the frame- D' form of $\mathcal{M}_P$ is $\text{swap}(CP \otimes PC)$, while $\text{swap}(CP \otimes CP)$ is $\mathcal{M}_P \circ (-1)^{F'}$ (Sec. V C).

Element	Twist	P	Order mod $W \times W'$	X^2 mod $W \times W'$	Frame D / D'	Chiral matter	Helicity	$\epsilon \neq 0$
$\mathcal{M}$	same	no	4	$C \otimes C$	$\text{swap}(C \otimes 1) / \text{swap}(1 \otimes C)$	no	kept	no
$\mathcal{M}_P$	same	yes	2	1	$\text{swap}(P \otimes P)$ [mirror] / $\text{swap}(CP \otimes CP)$ [copy]	yes	flipped	no
$\tilde{\mathcal{M}}$	mixed	no	2	1	swap [copy] / $\text{swap}(C \otimes C)$ [mirror]	yes	kept	yes
$\tilde{\mathcal{M}}_P$	mixed	yes	4	$C \otimes C$	C on one sector only	no	flipped	yes

are discussed in Sec. V.

The mirror map of Ref. [10] is the composition $P_M \Gamma$ of an exchange of the sectors with a chirality operator Γ that flips the sign of the left-handed fields. On the Standard Model content of one sector, $\Gamma = e^{6\pi i Y}$, the central element $-1_2 \in SU(2)_L$ of G_{SM} , which is trivial on the gauge bosons; on that content $P_M \Gamma$ is P_M up to a gauge transformation, and the Yukawa couplings are invariant. With the neutrinos shared between the sectors as in Ref. [10], Γ is not central in $G_{\text{SM}} \times G'_{\text{SM}}$ (it is gauge only up to the mirror fermion parity), and the invariance of the shared Yukawa coupling follows from the sign rule rather than from gauge invariance. The frame form $\text{swap}(C \otimes 1)$ of $\mathcal{M}$ in Eq. (24) is only an analogue of $P_M \Gamma$, not a swap up to gauge: C is not an element of G_{SM} and not a symmetry of the chiral Standard Model, and $\mathcal{M}^2 = C \otimes C \neq 1$.

In short, the map between the sectors that acts on chiral matter is $\mathcal{M}_P$, which in frame D is mirror parity; the twist forced by the self-duality does not produce the helicity flip, and what distinguishes mirror from copy matter is the orientation character. The chirality exchange of the two-dimensional model of App. C, in contrast, does not carry over to a four-dimensional helicity exchange (App. C 4).

F. Two-dimensional analogue: T-duality

The two-dimensional mirror models of Ref. [10] relate the mirror map to T-duality. To compare, we build the analogues of $\mathcal{M}$ and $\mathcal{M}_P$ in two dimensions, where the role of S is played by T-duality [46]. We take one compact boson per sector, $X^i \sim X^i + 2\pi$ ($i = 1$ ordinary, $i = 2$ mirror), on a common worldsheet,

$$\mathcal{L} = \frac{1}{4\pi} [G_{ij} \eta^{ab} \partial_a X^i \partial_b X^j + \epsilon^{ab} B_{ij} \partial_a X^i \partial_b X^j], \quad (44)$$

with the background $E = G + B = \begin{pmatrix} G_{11} & G_{12} + b \\ G_{12} - b & G_{22} \end{pmatrix}$, in which G_{12} is the analogue of kinetic mixing and b that of θ_{mix} . Worldsheet parity Ω ($\sigma \rightarrow -\sigma$) acts as $E \rightarrow E^T$ and lies outside the duality group $O(2, 2; \mathbb{Z})$ [46].

For one boson, T-duality $R \rightarrow 1/R$ has $T^2 = 1$ but $(\Omega T)^2 = C$. The two-dimensional versions of the mirror maps are $M_{2D} = \text{swap}(T \otimes T)$, of order 2, and $M_{P,2D} = \text{swap} \Omega(T \otimes T)$, of order 4, with lifts $\tilde{M}_{2D} = M_{2D}(1 \otimes C)$ and $\tilde{M}_{P,2D} = M_{P,2D}(1 \otimes C)$. The fixed loci of M_{2D} and $M_{P,2D}$ are $\{G_{12} = 0, G_{11}G_{22} = 1 - b^2\}$ and $\{G_{12} = b = 0, G_{11}G_{22} = 1\}$; those of $\tilde{M}_{2D}$ and $\tilde{M}_{P,2D}$ are $\{b = 0, \det G = 1\}$ and $\{\det G + b^2 = 1\}$, so at $b = 0$ the lifts fix $\det G = 1$ with G_{12} free; and in the decoupled limit $G_{12} = b = 0$ every locus reduces to $R_1 R_2 = 1$, the analogue of Eq. (21). The maps generate $\langle M_{2D}, M_{P,2D} \rangle \cong D_4$, with center $\{1, C \otimes C\}$. If we identify the winding and momentum numbers (w, n) with (n_m, n_e) , which is a choice and not the result of a reduction, then $\Omega = P$ and $T = CPS$, and

$$M_{2D} = \mathcal{M}_P(C \otimes C), \quad M_{P,2D} = \mathcal{M}^{-1}, \quad (45)$$

so the two groups coincide, but with the orientation character exchanged: the $\mathbb{Z}_4$ preserves orientation in four dimensions, where $S^2 = C$, and needs worldsheet parity in two, where $(\Omega T)^2 = C$. The reason is that the four-dimensional charge pairing is antisymmetric while the two-dimensional one, the spin $w \cdot n$, is symmetric. Reference [47] relates the Montonen–Olive duality of $\mathcal{N} = 4$ Yang–Mills theory on a two-torus to the T-duality of a two-dimensional lattice theory; the reduction of our four-dimensional elements is the subject of Sec. VID, and App. C applies these maps to a model with charged supersymmetry.

The two-dimensional mirror models SMM2 and SMM2b of Ref. [10] can be read in this setting. We take each sector to be the $\mathcal{N} = (1, 1)$ compact multiplet of SMM2b, with supercurrents $G = \lambda \partial \phi_L$ and $\bar{G} = \bar{\lambda} \bar{\partial} \phi_R$, and replace the bosonic gauge-field term of SMM2 by the free compact-boson action of Eq. (44); this replacement is our modeling choice. Each sector then has $c = \bar{c} = \frac{3}{2}$, and the two sectors together have $c = 3$, the value Ref. [10] requires in $D = 2$. T-duality commutes with supersymmetry only if it acts on the fermions as $\lambda \rightarrow \lambda, \bar{\lambda} \rightarrow -\bar{\lambda}$; then $T^2 = \Omega^2 = 1$ and $(\Omega T)^2 = C$. M_{2D} has order 2 and preserves the total supercurrents $(G, \bar{G})_{\text{tot}}$, while $M_{P,2D}$ has order 4 and exchanges G_{tot} and $\bar{G}_{\text{tot}}$. A potential $V(\Phi)$ for the multiplet Φ is excluded: invariance

of the Yukawa term $V''(\phi)\lambda\bar{\lambda}$ under T forces $V'' = 0$, and T commutes with supersymmetry only at vanishing auxiliary field, which forces $V' = 0$. The maps are thus symmetries only of the free compact theory, at $R = 1$ for one sector or on the fixed loci above for two; the Liouville potential of Ref. [10] is excluded.

G. What self-duality adds

The question of this paper is whether the self-duality of G_{SM} helps to build the mirror map. The maps (18) themselves do not need a self-dual gauge group: for any G , $\text{swap} \circ (S \otimes S)$ maps the family $T[G, \tau] \otimes T[G^\vee, \tau']$ to itself. To isolate what the self-duality of G_{SM} contributes, we compare with the global forms $G_k = (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_k$, $k \in \{1, 2, 3\}$, with $\mathbb{Z}_k$ inside the $\mathbb{Z}_6$ of Eq. (6), which carry the same matter but are not self-dual, and with the pair $SU(N)$, $PSU(N)$. As in the rest of this section the comparison is made at the level of lattices, with S formal; S is the canonical map $(m, e) \mapsto (e, -m)$ in the labels of G , acting on all factors at once. We call a map of a factor, or of a sector, *S-type* if it is the canonical S followed by an identification of root data, i.e. a map that sends roots to coroots, at the origin of the θ angles (without T shifts), and *partial S-type* if it acts in this way on some factors and as ± 1 on the others. Dualities that include T shifts are discussed at the end of this subsection.

For $k < 6$, $\pi_1(G_k) = \mathbb{Z}$ while $\pi_1(G_k^\vee) \cong \mathbb{Z} \oplus \mathbb{Z}_{6/k}$, so $G_k^\vee$ is none of the G_j : $G_1^\vee = PSU(3) \times SO(3) \times U(1)$, $G_2^\vee = PSU(3) \times U(2)$ and $G_3^\vee = U(3) \times SO(3)$, while $G_6^\vee \cong G_6 = G_{\text{SM}}$. Root-datum isomorphisms from G_j onto $G_k^\vee$ exist only for $j = k = 6$; they are the 24 of Sec. II A, and in the labels of Sec. II C each is either a color conjugation with $y \mapsto +\tilde{Y}$ or a Weyl element on $SU(3)$ with $y \mapsto -\tilde{Y}$, twelve of each; this is the twist of Sec. II D. Among representations of Standard Model type, color singlets or triplets and $SU(2)$ singlets or doublets, the electric frame of $G_k^\vee$ admits color triplets only for $k = 3, 6$ and $SU(2)$ doublets only for $k = 2, 6$; for $k = 1$ it admits only **(1, 1)**.

In coupling space the product $T[G_k, \tau] \otimes T[G_k^\vee, \tau']$ behaves like $G_{\text{SM}} \times G'_{\text{SM}}$ for every k : $\mathcal{M}$ and $\mathcal{M}_P$ preserve its line lattice, act on each factor as in Eq. (19) with the fixed loci (20), obey Eq. (23), map the roots of each sector to the coroots of the other, and take the frame forms (24) with $D = 1 \otimes S$, which maps the product onto two identical copies of $T[G_k]$. What fails for $k < 6$ is the factorization: neither the swap nor any per-factor $S \otimes S$ with signs preserves the lattice $L(G_k) \oplus L(G_k^\vee)$, only their composites do; and on identical copies $G_k \times G_k$ the swap is a symmetry, but no element $\text{swap}(S \otimes S)$ with S on all three factors is. For G_{SM} , the swap and $S_F \otimes S_F$ are separately symmetries of the two-sector lattice. The failure holds in both the product frame and the frame D : a sector swap and an S-type self-map of each sector can co-exist only if there is a root-datum isomorphism $G_k \cong G_k^\vee$, and there is none for $k = 1, 2, 3$, while there are 24 for

$k = 6$.

Within one sector, for $k < 6$ no S-type map on all three factors, with any signs, takes $L(G_k)$ to the lattice of any G_j , or to a θ shift of one, so a single G_k sector has no such self-duality and no self-dual point; for G_{SM} , S_F fixes $\tau_3 = \tau_2 = \hat{\tau}_Y = i$ (Sec. IIIB). Partial S-type self-dualities do exist: $\hat{\tau}_Y \mapsto -1/(36\hat{\tau}_Y)$ for G_1 ; $\tau_2 \mapsto -1/\tau_2$ with $\hat{\tau}_Y \mapsto -1/(9\hat{\tau}_Y)$ for G_2 ; and $\tau_3 \mapsto -1/\tau_3$ with $\hat{\tau}_Y \mapsto -1/(4\hat{\tau}_Y)$ for G_3 , so each G_k sector has a partial self-dual locus. The canonical S carries no twist and acts as $\hat{\tau}_Y \mapsto -1/\hat{\tau}_Y$ for every k , so $g_Y g'_Y = 12\pi$ keeps its form, but only in labels inherited from G_k : the mirror sector then carries no Standard Model hypercharge. If instead the mirror $U(1)$ is normalized, like the hypercharge of G_{SM} , so that the smallest $U(1)$ charge of any representation of $G_k^\vee$ is $\frac{1}{6}$, the relation at $\theta = 0$ becomes $g_Y g'_Y = 72\pi/k$, i.e. 72π , 36π and 24π for $k = 1, 2, 3$, and 12π for $k = 6$; for $k = 1$ this is the value 72π of the cover found in Sec. IIIB. Normalizing instead by the charges of the pure $U(1)$ sublattice gives $g_Y g'_Y = 72\pi$ for every k . The lattices of θ periods of the two sectors also differ for $k < 6$, whereas for G_{SM} both are the lattice Λ_θ of Eq. (36). In the labels of $G_{\text{SM}}^\vee$, finally, the image $S(L(G_{\text{SM}}))$ is $L(G_{\text{SM}})$ relabeled by color conjugation, or equally by $y \rightarrow -y$: the twist forced by the glue (Sec. IIID) is the price of identifying $G_{\text{SM}}^\vee$ with G_{SM} .

For line operators, $\mathcal{M}$ and $\mathcal{M}_P$ send every Wilson line $W(e)$ of G_k to a mirror 't Hooft line $H'(\pm e)$, and the Standard Model Wilson lines to the free flux class with $|\tilde{Y}'| = |y|$, as for G_{SM} [Eq. (55)]. The Wilson lines of G_k that the Standard Model matter leaves unscreened go to mirror 't Hooft lines with a nonzero torsion component in $\pi_1(G_k^\vee) \cong \mathbb{Z} \oplus \mathbb{Z}_{6/k}$, which no Standard Model image has. A mirror copy of Standard Model type, with color triplets and $SU(2)$ doublets, can be electric in the electric frame of $G_k^\vee$ only for $k = 6$. For every k the rule “ordinary electric $\leftrightarrow$ mirror magnetic” is forced by the matter, since the images of the Standard Model Wilson lines carry the free flux $(\pm y, 0)$, with $\gcd|y| = 1$, which electric mirror matter cannot screen (Sec. IV B); for $k < 6$ it is forced by the gauge group as well, since $G_1^\vee$ and $G_2^\vee$ admit no electric color triplets and $G_1^\vee$ and $G_3^\vee$ no electric $SU(2)$ doublets.

The pair $SU(N)$, $PSU(N)$, $N = 2, \dots, 6$, shows the same pattern without an abelian factor. The canonical S maps the lattice of $SU(N)$ onto that of $PSU(N)_0$, with p in $PSU(N)_p$ the discrete θ label; $\mathcal{M}$ and $\mathcal{M}_P$ preserve the lattice of $T[SU(N), \tau] \otimes T[PSU(N)_0, \tau']$ and obey Eq. (23), and $D = 1 \otimes S$ gives $SU(N) \times SU(N)$. There is no twist. As $PSU(N)$ has no electric charge of nonzero N -ality, the mirror matter must be magnetic by the gauge group alone.

The failures above concern S-type maps. Dualities that include T shifts give more. For $k = 3$ the map

$$\gamma_3 : \tau_3 \mapsto -\frac{1}{\tau_3}, \quad \tau_2 \mapsto \frac{1 - \tau_2}{1 - 2\tau_2}, \quad \hat{\tau}_Y \mapsto -\frac{1}{4\hat{\tau}_Y} \quad (46)$$

acts on all three factors, preserves $L(G_3)$ and has or-

TABLE III. Statements of Secs. III A–III D and IV for $G_{\text{SM}} \times G'_{\text{SM}}$ and for the non-self-dual products $T[G_k] \otimes T[G_k^\vee]$, $k \in \{1, 2, 3\}$, with the canonical S on all factors, at the level of lattices. “S-type” maps are S followed by a root-datum identification, without T shifts; the entries hold in the product frame and in the frame D . The two rows on S-type maps restate whether a root-datum isomorphism $G \cong G^\vee$ exists; the row with T shifts refers to Eq. (46). The value of $g_Y g'_Y$ is for the mirror $U(1)$ normalized to the smallest charge $\frac{1}{6}$ of any representation of $G_k^\vee$. The list is not exhaustive: the twist classes of Table II, the 1-form, flavor and non-invertible symmetries and the selection of the global form were not compared.

Statement	$G_{\text{SM}} \times G'_{\text{SM}}$	$G_k \times G_k^\vee$
Eqs. (20), (23), (24)	yes	yes
$W_G \mapsto H_{G'}$	yes	yes
Swap and S-type self-maps coexist	yes	no
S-type self-dual point, all factors	yes	no
Order-4 self-duality on all factors, T shifts allowed	yes	$k = 3$ only
Common θ -period lattice	yes	no
$g_Y g'_Y$ at $\theta = 0$	12π	$72\pi/k$
Magnetic mirror forced by C-twist	matter forced	matter, group absent

der 4, $\gamma_3^2 = C$. It fixes $(\tau_3, \tau_2, \hat{\tau}_Y) = (i, \frac{1}{2} + \frac{i}{2}, \frac{i}{2})$, i.e. $\theta_2 = \pi$ and $g_2^2 = 8\pi$, and sends the $SU(2)$ W boson to the dyon $(n_m, n_e) = (2, -1)$. In the frame D both the swap and $\gamma_3 \otimes \gamma_3$ preserve $L(G_3) \oplus L(G_3)$, so for $k = 3$, in that frame, a sector swap and an order-4 self-duality of each sector coexist once T shifts are allowed. For $k = 1, 3$ the $SU(2)$ factor splits off, with duality group $\Gamma_0(2)$ ($\tau \mapsto (a\tau + b)/(c\tau + d)$, c even), whose order-4 elements, such as the $SU(2)$ part of γ_3 , are self-dualities that are not S-type. This $SU(2)$ element is the TST^{-1} of the Seiberg–Witten construction of App. B 2. For $k = 1, 2$, in contrast, no order-4 map with an element of $SL(2, \mathbb{Q})$ on each factor acts on all three factors (maps that mix the factors are not considered), because the self-maps of the $SU(3)$ factor lie in $\Gamma_0(3)$, which has no element of order 4; the reason is thus not the absence of a root-datum isomorphism. We do not count lattice maps of Fricke type, $\tau_2 \mapsto -1/(4\tau_2)$ and its $SU(3)$ analogue $\tau_3 \mapsto -1/(3\tau_3)$ combined with a reflection of the Cartan algebra: they preserve the lattices but send W bosons to magnetic charges that are not coroots.

Table III collects the comparison. The fixed loci, the abstract group structure of $\mathcal{M}$ and $\mathcal{M}_P$ and the exchange of Wilson and 't Hooft lines do not depend on self-duality. What self-duality adds are three things: the electric-frame group of the mirror sector is G_{SM} , the two sectors share one lattice of θ periods, and the dictionary carries the C-twist, which is its price. The first two “no” entries of the table restate the absence of a root-datum isomorphism $G_k \cong G_k^\vee$, since an S-type map identifies the root data by definition, and once T shifts are included

the form with $k = 3$ also has an order-4 self-duality on all three factors. That the mirror matter is magnetic is forced by the matter in every case, and without self-duality by the gauge group as well. For mirror matter, then, self-duality is what allows the mirror sector to be a Standard Model sector in its own electric frame, while the coupling relations and the line map do not depend on it.

H. Frame-independent content

Whether $\mathcal{M}_P$ is a physical symmetry depends on S, which we cannot settle, and many statements above depend on the frame in which the mirror sector is described. This subsection separates what survives in every case. Two cases must be distinguished. If per-sector S is an exact equivalence, $\mathcal{M}_P$ is conjugate by $D = 1 \otimes S$ to swap($P \otimes P$), Eq. (24), and its fixed locus maps onto the exact mirror-parity locus $g'_D = g$, $\theta'_D = -\theta$ (Sec. V A). The theory then has the spectrum and observables of conventional mirror matter, described in a magnetic frame for the mirror sector, and $\mathcal{M}_P$ is the standard mirror parity in a dual frame. If per-sector S is not an exact equivalence, as the running couplings, the $B + L$ anomaly and the absence of smooth monopoles indicate for the chiral Standard Model (App. B), $\mathcal{M}_P$ is not a symmetry, and its fixed locus can hold at most at one scale (Sec. V B).

In either case a statement that refers to the electric frame of the mirror sector depends on the frame. This includes the coupling relations (27) and their form $\alpha_3 \alpha'_3 = \alpha_2 \alpha'_2 = 1$, $\alpha_Y \alpha'_Y = 9$ in Sec. V, which in the frame D become $g'_D = g$; the assignment of the constraint to ϵ or to θ_{mix} , since at $\theta_Y = 0$ the element $\mathcal{M}_P$ forces $\epsilon = 0$ in the original frame [Eq. (41)] but leaves ϵ_D free and forces the mixed angle of the frame D to vanish [Eq. (65)]; and the labels “mirror” and “copy” of the second sector, which are exchanged between the frames D and D' (Table II).

The following statements do not depend on the frame. (i) The group theory: G_{SM} is self-dual, it is the only self-dual quotient of $SU(3) \times SU(2) \times U(1)$ by a subgroup of $\mathbb{Z}_6$, and among the 600 line lattices of Sec. IV B only $L(G_{\text{SM}}) \oplus L(G'_{\text{SM}})$ is mapped by $\mathcal{M}$ or $\mathcal{M}_P$ to a lattice that contains the matter of both sectors. (ii) The line dictionary $W_G \leftrightarrow H_{G'}$ of Eq. (54) and its consequence for the matter: $\mathcal{M}$ or $\mathcal{M}_P$ can be a symmetry with matter only if the mirror matter is magnetic with respect to the mirror’s electric frame, i.e. electric in the frame D . (iii) The group structure (23), which conjugation by D preserves, and the orientation character: an element compatible with chiral matter either reverses orientation and flips helicity ($\mathcal{M}_P$, $\theta' = -\theta$ in the frame D) or preserves both ($\tilde{\mathcal{M}}$, $\theta' = \theta$), and this, not the “mirror” or “copy” label, is the invariant. (iv) Statements that use no duality at all: which classical symmetries force the kinetic mixing to vanish, and the radiative stability of $\epsilon = \theta_{\text{mix}} = 0$ in the absence of Higgs-portal couplings and of states

charged under both hypercharges (Sec. VD). The comparison of Sec. III G is not in this list: its statements about S-type maps were established in the product frame and in the frame D , not in every frame.

Physical θ -angles. What the loci with $\theta \neq 0$ say about the invariants (38) depends on the same two cases. Suppose first that the maps act on the gauge couplings only, with the matter of both sectors electric in the original frame, so that each sector's chiral rotations act as real shifts of its own τ 's together with its Yukawa phases. Then $\mathcal{M}_P$ does not commute with these reparametrizations: conjugating a shift $\tau \mapsto \tau + x$ of the ordinary coupling by $\mathcal{M}_P$ gives $\tau' \mapsto \tau'/(1 + x\tau')$, which changes $\text{Im } \tau'$, so $\text{Fix}(\mathcal{M}_P)$ is not a union of their orbits; the same holds for $\tilde{\mathcal{M}}_P$. The only constant slice $\{\theta = a, \theta' = a'\}$ that $\mathcal{M}_P$ maps to itself for all couplings is $a = a' = 0$, the slice of vanishing removable angles of Sec. III D; on the hypercharge block this holds only where $\epsilon \theta_{\text{mix}} = 0$. On this slice $g_k g'_k = 4\pi$ for $k = 2, 3$, and $\bar{\theta}, \bar{\theta}'$ are unconstrained, since they sit in the Yukawa phases. The slice of real Yukawa matrices, on which $\theta_3 = \bar{\theta}$ and $\theta_2 = \theta'_2 = 0$ by $B + L$, is also invariant, since the maps do not act on the Yukawa couplings, but there the locus (40) reads

$$\bar{\theta}' g_3'^2 = \bar{\theta} g_3^2, \quad \frac{\bar{\theta}\bar{\theta}'}{4\pi^2} + \frac{16\pi^2}{(g_3 g_3')^2} = 1, \quad (47)$$

i.e. $\bar{\theta}' = \alpha_3^2 \bar{\theta} + O(\bar{\theta}^3)$. The two slices give different physical loci, so in this reading $\text{Fix}(\mathcal{M}_P)$ constrains $\bar{\theta}$ and $\bar{\theta}'$ only through a choice of slice. Eliminating the removable angles altogether leaves, with $\bar{\theta}$ and $\bar{\theta}'$ free,

$$g_k g'_k \geq 4\pi, \quad g_Y g'_Y \geq \begin{cases} 12\pi(1 + |\epsilon|) & (\mathcal{M}_P), \\ 12\pi\sqrt{1 - \epsilon^2} & (\tilde{\mathcal{M}}_P), \end{cases} \quad (48)$$

for $k = 2, 3$; off the slice, $\epsilon \neq 0$ is allowed on $\text{Fix}(\mathcal{M}_P)$.

Suppose instead that per-sector S is an exact equivalence. The mirror matter is then electric in the frame D , and the mirror angle that combines with the mirror quark masses is $\theta'_{3,D} = 2\pi \text{Re}(-1/\tau'_3)$, not θ'_3 . On $\text{Fix}(\mathcal{M}_P)$, $\tau'_D = -\bar{\tau}$, i.e. $\theta'_D = -\theta$ and $g'_D = g$ [Eq. (65)]. In the original frame this is the pair of equations (40); the first alone, $\theta'_3 g_3'^2 = \theta_3 g_3^2$, is not equivalent to $\theta'_{3,D} = -\theta_3$. To act on the matter, $\mathcal{M}_P$ must also be extended to the Yukawa couplings, and S does not fix this extension. We extend it by parity, mapping the left-handed term $Y\psi\psi H$ to the right-handed mirror term with the same coefficient, so that $\varphi'_f = \varphi_f$; among affine maps of the phases φ_f alone this is the only extension that commutes with the reparametrizations of both sectors. Its fixed set is

$$\bar{\theta}' = -\bar{\theta} \pmod{2\pi}, \quad I'_{\text{EW}} = -I_{\text{EW}} \pmod{4\pi}, \quad (49)$$

together with $g'_D = g$, $\theta_{\text{mix},D} = 0$ and ϵ_D free. Here both angles are measured in a common orientation of the two sectors. The sign depends on the type of exchange symmetry: copy-type exchange symmetries, which map

left-handed fermions to left-handed mirror fermions, give $\bar{\theta}' = +\bar{\theta}$ [48, 49], while parity-type ones give the opposite sign, as in Eq. (49): the mass-matrix phases of the two chiralities in Ref. [50] [its Eq. (3)] are opposite, and parity solutions carry “the opposite theta angle” in the opposite-parity sector [49]. This is the orientation character of item (iii): a copy-type $\mathbb{Z}_2$ corresponds to the helicity-preserving class $\tilde{\mathcal{M}}$, with $\theta' = \theta$, and in the frame D' , with copy content, the CP extension of $\mathcal{M}_P$ gives the same relation as in D . The mirror sector therefore has a strong-CP problem of the same size, $|\bar{\theta}'| = |\bar{\theta}|$, which $\mathcal{M}_P$ neither removes nor relates to zero. The element $\tilde{\mathcal{M}}_P$, which is not compatible with chiral matter, has no such extension and constrains θ' only through a slice, as above. Both readings rest on the counting of Eq. (38), with one Higgs doublet per sector.

In the frame D , where $\mathcal{M}_P$ is $\text{swap}(P \otimes P)$, Eq. (49) is the relation that the standard mirror parity imposes on the two vacuum angles, since $\bar{\theta}$ is odd under parity; the present construction adds only its reading in the original frame, where the mirror angle that enters is $\theta'_{3,D}$ rather than θ'_3 , and the condition that S be exact. Parity relating left-handed fermions to right-handed partners is the basis of the parity solutions of the strong-CP problem, those of Refs. [51, 52], reviewed in Ref. [53], and that of Ref. [50], in which one color group is shared and the relation forces $\bar{\theta} = 0$ without an axion. With two color groups, as here, the relation ties $\bar{\theta}'$ to $\bar{\theta}$ and leaves their common size free. Mirror sectors with their own color group and a copy-type exchange symmetry have also been used to address the strong-CP problem, through one Peccei–Quinn axion coupled to both sectors [48], or through a $\mathbb{Z}_2$ copy of QCD together with an anomalous $U(1)$ or massless bifundamental quarks, without an axion [49]; we do not consider such additions.

IV. LINE OPERATORS AND GENERALIZED SYMMETRIES

The self-duality of G_{SM} is a property of the gauge group. It does not by itself guarantee that a duality maps the full theory, with its matter, to itself: the global form, the matter content and the θ -angles together fix which line operators are genuine [13, 54], and with them the generalized global symmetries of the theory [55, 56]. Line operators are also the most direct way to see what the mirror maps do physically: a Wilson line is the world-line of a heavy electric charge and an 't Hooft line that of a heavy monopole, so the map of lines tells which particle of one sector corresponds to which object of the other. In this section we write the lattice of Wilson and 't Hooft lines of G_{SM} with Standard Model matter, map it with $\mathcal{M}$ and $\mathcal{M}_P$, and follow the 1-form, flavor and non-invertible symmetries through the map. As in Sec. III, S is a formal map on charge lattices and couplings. All statements are made at the level of charge lattices and anomaly coefficients, on spin four-manifolds, with line labels at the

origin of the θ -angles. We treat the 1-form symmetries as exact, although the magnetic one is accidental in the Standard Model [57, 58]. Portal operators between the two sectors are not included.

A. One sector

A line operator of a gauge theory with group G carries a magnetic charge $m \in X_*$ and an electric charge $e \in X^*$, each defined up to the Weyl group [13, 33]. We write Wilson lines as $W(a_1, a_2, n, y)$ and 't Hooft lines as $H(b_1, b_2, n^\vee, \tilde{Y})$, in the coordinates of Eq. (7), and a line is screened if it can end on a dynamical particle (matter or gauge bosons); there are no dynamical monopoles. The weights of one Standard Model generation span $X^* = \Lambda_e$, so the magnetic charges that are Dirac compatible with the matter are exactly $X_* = \Lambda_m$, and the lattice of genuine lines is

$$L = X^* \oplus X_*, \quad (50)$$

which is unimodular under the pairing (8). Wilson lines form $X^* \oplus 0$, 't Hooft lines $0 \oplus X_*$, and all other elements are dyonic. Every Wilson line is screened. Modulo the dynamical charges and the coroot 't Hooft charges $Q^\vee$, the lines fall into the classes

$$L/(X^* \oplus Q^\vee) \cong \mathbb{Z}, \quad (51)$$

measured by $\tilde{Y}$, the flux class in $\pi_1(G_{\text{SM}}) \cong \mathbb{Z}$.

The θ -angles shift the electric labels of lines with magnetic charge [40]. With the θ -terms of App. A, a $U(1)$ line (n_m, n_e) carries the electric charge $e(n_e + \theta n_m/2\pi) = e \operatorname{Re} Z$, $Z = n_m \tau + n_e$. For G_{SM} the shift is block diagonal,

$$\begin{aligned} \Delta(a_1, a_2) &= \frac{\theta_3}{2\pi}(b_1, b_2), & \Delta n &= \frac{\theta_2}{2\pi} n^\vee, \\ \Delta Y &= \frac{\theta_Y}{4\pi} \tilde{Y}, \end{aligned} \quad (52)$$

and the θ shifts that preserve L are exactly the periods Λ_θ of Eq. (36). Like the periods, Eq. (52) treats the θ -angles as couplings in a fixed fermion basis (Sec. III D).

The other global forms behave differently. For $G_k \equiv (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_k$, with $\mathbb{Z}_k$ the subgroups of the $\mathbb{Z}_6$ of Sec. II B and $Y(Q_L) = \frac{1}{6}$ fixed, there are 12 line lattices with electric labels in the lattice of the cover and magnetic labels in Λ_m : 1, 2, 3 and 6 of them for $k = 1, 2, 3$ and 6, the variants with the same k being the same global form at shifted θ [13]. The Wilson lines of G_k modulo screening form $\mathbb{Z} \oplus \mathbb{Z}_{6/k}$. For all 24 isomorphisms Φ , S_Φ maps L onto itself and exchanges its Wilson and 't Hooft sublattices; for every other form, and every Φ , the image of the line lattice does not contain the Standard Model Wilson lines.

a. 1-form symmetries. The electric 1-form symmetry acts on Wilson lines through the center elements that act trivially on the matter, and the magnetic one is the Pontryagin dual of π_1 [55, 56]. In the pure gauge theory with group G_k they are $\Gamma_e = U(1) \times \mathbb{Z}_{6/k}$ and $\Gamma_m = U(1)$, since $\pi_1(G_k) \cong \mathbb{Z}$ is torsion free for every k . With one generation of matter, $\Gamma_e = \mathbb{Z}_{6/k}$, which is trivial for G_{SM} , and $\Gamma_m = U(1)$ is unchanged. With matter, the only 1-form symmetry of G_{SM} is therefore $U(1)_m^{(1)}$, with charge $\tilde{Y}$. It has no 't Hooft anomaly, and since the periods Λ_θ leave $\tilde{Y}$ invariant, there is no mixed anomaly with the periodicity of the θ -angles. In the pure gauge theory the electric $U(1)_e$, generated by the hypercharge cocharacter λ_Y and with charge y on Wilson lines, and $U(1)_m$ have the mixed anomaly

$$\frac{ik}{2\pi} \int \mathcal{B}_e \wedge d\mathcal{B}_m, \quad k = \langle \ell_Y, \lambda_Y \rangle = 6, \quad (53)$$

where $\mathcal{B}_e, \mathcal{B}_m$ are the background 2-form fields and ℓ_Y is the functional $\tilde{Y}$ of Eq. (34); a subgroup $\mathbb{Z}_N \subset U(1)_e$ is free of it if and only if $N \mid 6$.

The $\mathbb{Z}_3$ magnetic symmetry of the $\mathbb{Z}_3$ and $\mathbb{Z}_6$ quotients noted in Ref. [58] (footnote 9) is, in our coordinates, the subgroup $\{e^{2\pi i j \tilde{Y}/3}\}$ of $U(1)_m$, not torsion in π_1 . For $k < 6$ with matter, the electric $\mathbb{Z}_{6/k}^{(1)}$ has a mixed anomaly of order $6/k$ with $U(1)_m$, and gauging it gives G_{SM} .

B. Two sectors

For $G_{\text{SM}} \times G'_{\text{SM}}$ with one generation in each sector, the line lattice is $(X^* \oplus X_*) \oplus (X'^* \oplus X'_*)$. By Table II, only $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ are compatible with chiral matter, while $\mathcal{M}$ and $\tilde{\mathcal{M}}_P$ are not. In this section the statements about $\mathcal{M}$, including those on 1-form symmetries [Eq. (57)], flavor anomalies [Eqs. (58) and (60)] and non-invertible defects, are therefore statements about lattices and anomaly coefficients, given for comparison; only $\mathcal{M}_P$ (and $\tilde{\mathcal{M}}$) can act on the chiral matter. With the mixed angle θ_{mix} , an ordinary line with hypercharge flux $\tilde{Y}$ carries the mirror hypercharge $\Delta Y' = \theta_{\text{mix}} \tilde{Y}/4\pi$, and vice versa; the lattice is preserved by $\Delta \theta_{\text{mix}} \in 4\pi\mathbb{Z}$, as in Sec. III D. For all 24×24 pairs (Φ, Φ') , the elements $\mathcal{M}$ and $\mathcal{M}_P$ map the ordinary Wilson lattice X^* onto the mirror 't Hooft lattice X'_* , the ordinary 't Hooft lattice onto the mirror Wilson lattice, and the mirror lattices back onto the ordinary ones in the same way:

$$\mathcal{M}, \mathcal{M}_P : \quad W_G \mapsto H_{G'}, \quad H_G \mapsto W_{G'}. \quad (54)$$

Dyonic lines go to dyonic lines, and the Coulomb energy of every line is preserved, including the contributions of kinetic mixing and θ_{mix} . The image of an ordinary Wilson line depends only on Φ , and $\mathcal{M}_P$ reverses its orientation relative to $\mathcal{M}$. A Wilson line of hypercharge $y = 6Y$ goes to a mirror 't Hooft line with hypercharge flux class

$$\tilde{Y}' = \mp y \quad (\mathcal{M}), \quad \tilde{Y}' = \pm y \quad (\mathcal{M}_P), \quad (55)$$

TABLE IV. Images of the Wilson lines $W(a_1, a_2, n, y)$ of one Standard Model generation and of the Higgs doublet under $\mathcal{M}_P$ with Φ in twist class A, given by the dominant labels $(b_1, b_2, n^\vee, \tilde{Y})$ of the pure mirror 't Hooft lines H' . Under $\mathcal{M}$ the lines have the opposite orientation (the magnetic duals of Table I); class B reverses the sign.

Field	$W(a_1, a_2, n, y)$	$\mathcal{M}_P$ image $H'(b_1, b_2, n^\vee, \tilde{Y})$
Q_L	(1, 0, 1, 1)	(0, 1, 1, 1)
u_R	(1, 0, 0, 4)	(0, 1, 0, 4)
d_R	(1, 0, 0, -2)	(0, 1, 0, -2)
L_L	(0, 0, 1, -3)	(0, 0, 1, -3)
e_R	(0, 0, 0, -6)	(0, 0, 0, -6)
H	(0, 0, 1, 3)	(0, 0, 1, 3)

with the upper sign for Φ in twist class A (which contains F) and the lower one for class B (Sec. III E); roots go to coroots.

This has a consequence for the matter. Since $|\tilde{Y}'| = |y|$ and the values of $|y|$ in one generation have greatest common divisor 1, the images of the ordinary dynamical charges reach every class of $L'/(X^{*\vee} \oplus Q'^\vee) \cong \mathbb{Z}$, and none of these classes is screened in the electric description of the mirror sector. Hence $\mathcal{M}$ or $\mathcal{M}_P$ can be a symmetry of the theory with matter only if the dynamical matter of the mirror sector is magnetic in the mirror's electric frame. This is the precise form of "ordinary electric $\leftrightarrow$ mirror magnetic"; it is a necessary condition. Chiral fermions with magnetic charges are constrained by anomaly conditions of their own [59], and low-energy abelian theories with electric and magnetic charges transform covariantly under $SL(2, \mathbb{Z})$ [60]. The global form is also fixed: of the 600 line lattices with labels in the electric lattices of the covers and in the magnetic lattices Λ_m, Λ'_m of both sectors (the Lagrangian subgroups of $\mathbb{Z}_6^4$, including those that mix the sectors), only $L(G_{\text{SM}}) \oplus L(G'_{\text{SM}})$ at the origin of the θ -angles is mapped by $\mathcal{M}$ or $\mathcal{M}_P$ to a lattice that contains the matter of both sectors.

Each Standard Model Wilson line goes to a pure mirror 't Hooft line, unique up to W' within a twist class; the images are listed in Table IV. The electron is the simplest example. Its Wilson line $W(e_R)$, in the representation $(\mathbf{1}, \mathbf{1})_{-1}$, which becomes the Wilson line of electric charge -1 after electroweak symmetry breaking, is mapped as

$$\begin{aligned} W(e_R) &\mapsto H'(0, 0, 0, \mp 6) \quad (\mathcal{M}_P), \\ W(e_R) &\mapsto H'(0, 0, 0, \pm 6) \quad (\mathcal{M}), \end{aligned} \quad (56)$$

with the upper sign for Φ in class A and the lower one for class B, as in Eq. (55) with $y = -6$. This is the 't Hooft line of the mirror hypercharge cocharacter $\lambda_Y(t) = (t^3 \mathbf{1}_2, t^{-2} \mathbf{1}_3)$, with one of its two orientations. It is the minimal pure-hypercharge Dirac monopole line of $U(1)'_Y$: it carries no $SU(3)'$ or $SU(2)'$ flux, and its flux class in $\pi_1(G'_{\text{SM}}) \cong \mathbb{Z}$ is ± 6 . Conversely, the mirror electron's Wilson line $W'(e_R)$ goes to the ordinary 't Hooft line $H(0, 0, 0, \mp 6)$ under $\mathcal{M}_P$ and $H(0, 0, 0, \pm 6)$

under $\mathcal{M}$, now with the upper sign for Φ' in class A.

In the S-dual frame $D = 1 \otimes S_{\Phi'}$ of the mirror sector the picture is the familiar one: $\mathcal{M}_P$ (with Φ, Φ' in the same class) sends $W(e_R)$ to the mirror electron's Wilson line $W'(e_R)$, and $\mathcal{M}$ sends it to its conjugate. In this frame the mirror electron's Wilson line goes back to the ordinary Wilson line $W(e_R)$, for all 1152 elements. In the original frame the image is a monopole of the mirror hypercharge, not of the mirror photon. With $g'_Y \Phi_B = 2\pi \tilde{Y}$ for the flux Φ_B of B' , the line $H'(0, 0, 0, 6)$ (the image of $W(e_R)$ under $\mathcal{M}$ in class A) carries the photon flux $e' \Phi_A = 12\pi \cos^2 \theta'_W$ plus a Z' flux, which is confined after mirror electroweak symmetry breaking. The minimal color-singlet line with purely electromagnetic flux is the image of a Wilson line with $n = 3$ and $|y| = 3$, which is not a Standard Model representation.

a. 1-form symmetries. We label the 1-form symmetries by their charges, the functionals $f_y, f_{\tilde{Y}}, f_{y'}, f_{\tilde{Y}'}$ on the line lattice, which are transported with the lines, $f \mapsto f \circ X^{-1}$ for a duality X . We call $\mathcal{T}_{\text{el}}$ the theory whose matter is electric in the electric frames of both sectors, and $\mathcal{T}_D$ the theory whose mirror matter is magnetic in the mirror's electric frame, i.e. electric in the frame D . The pure gauge theory has $U(1)_e \times U(1)_m \times U(1)'_e \times U(1)'_m$, with charges $(y, \tilde{Y}, y', \tilde{Y}')$ and the anomaly (53) in each sector, $K(\tilde{Y}, y) = K(\tilde{Y}', y') = 6$, with no entries between the sectors; kinetic mixing does not enter. A shift $\Delta\theta_{\text{mix}} = 4\pi$ acts on a fixed line as the Witten effect $y \rightarrow y + 6\tilde{Y}', y' \rightarrow y' + 6\tilde{Y}$. With matter, $\mathcal{T}_{\text{el}}$ has the 1-form symmetry $U(1)_m \times U(1)'_m$, without anomaly and fixed by all θ periods. Every element of the form $\mathcal{M}$ or $\mathcal{M}_P$ acts as

$$\begin{aligned} f_y &\mapsto s_1 f_{\tilde{Y}'}, & f_{\tilde{Y}} &\mapsto s_2 f_{y'}, \\ f_{y'} &\mapsto s_3 f_{\tilde{Y}}, & f_{\tilde{Y}'} &\mapsto s_4 f_y, \end{aligned} \quad (57)$$

so ordinary electric charges go to mirror magnetic ones and ordinary magnetic charges to mirror electric ones, with signs $s_i = \pm 1$ fixed by the twist classes of Φ and Φ' and by $P[s_1]$ is Eq. (55) read on charges]. The anomaly goes to $K \rightarrow +K$ under $\mathcal{M}$ and to $K \rightarrow -K$ under the orientation-reversing $\mathcal{M}_P$.

The matching of 1-form symmetries then selects the theory. $\mathcal{M}$ and $\mathcal{M}_P$ map $\mathcal{T}_{\text{el}}$ to a theory with $U(1)_e \times U(1)'_e$, i.e. with dynamical monopoles in both sectors, so neither is a symmetry of $\mathcal{T}_{\text{el}}$. The theory $\mathcal{T}_D$ has the 1-form symmetry $U(1)_m \times U(1)'_e$, which every element maps onto itself, $f_{\tilde{Y}} \leftrightarrow \pm f_{y'}$, with matching (vanishing) anomalies. In the original frame the θ periods of one sector go to periods of the dual coupling of the other sector; in the frame D , every $D X D^{-1}$ maps $\Lambda_\theta \times \Lambda'_\theta \times 4\pi\mathbb{Z}$ onto itself, sending the generators of one sector to those of the other with the sign $(-1)^p$ ($p = 1$ for $\mathcal{M}_P$) and θ_{mix} to $\pm 4\pi$.

C. Flavor symmetries and 't Hooft anomaly matching

We now turn to the 0-form flavor symmetries. All anomaly coefficients below are group-theory factors of triangle diagrams; no loop is computed. For one generation without ν_R , the coefficients of the Adler–Bell–Jackiw (ABJ) anomalies $(\mathfrak{su}_3^2, \mathfrak{su}_2^2, Y^2)$ are $(0, \frac{1}{2}, -\frac{1}{2})$ for B and for L , $(0, 0, 0)$ for $B - L$ and $(0, 1, -1)$ for $B + L$, and with the Yukawa couplings $B - L$ is the only invertible flavor $U(1)$. Its 't Hooft anomalies are $(B - L)^3 = -1$ and $(B - L) \text{grav}^2 = -1$. The mirror sector, with the opposite chirality, has every coefficient reversed. Since orientation reversal also reverses 't Hooft anomalies, the anomalies match only for

$$\mathcal{M}_P : B - L \mapsto (B - L)', \quad \mathcal{M} : B - L \mapsto -(B - L)'. \quad (58)$$

For $\mathcal{M}$, which is not compatible with chiral matter, this is a statement about anomaly coefficients only (Sec. IV B). The flavor symmetries form no 2-group with $U(1)_m^{(1)}$: the coefficient $\kappa = \sum y X^2$ of Ref. [57], summed over the left-handed Weyl fields with charges X , vanishes for B , L and $B \pm L$, and $\sum y = 0$. In degree six the anomaly polynomial contains only F^3 , Fp_1 and products of field strengths of two different 2-form backgrounds, so $U(1)_m^{(1)}$ has no mixed 't Hooft anomaly with any continuous 0-form symmetry. Torsion anomalies [61], for example of discrete subgroups of $U(1)_{B-L}$, are not covered by this statement; for one generation $e^{i\pi(B+L)} = e^{i\pi(B-L)} = (-1)^F$ up to a gauge transformation.

With three generations the line operators do not change. The weights of $N_g = 1, 2$ and 3 generations, with or without H , span the same lattice X^* , and the results of Secs. IV A and IV B depend on the matter only through this span; they hold verbatim for $N_g = 3$, with $4N_g = 12$ Weyl doublets per sector and no Witten anomaly. The dualities act on charges, not on generation labels: the Wilson lines of e , μ and τ are one line, mapped to $H'(0, 0, 0, -6)$ by $\mathcal{M}_P$ and to $H'(0, 0, 0, +6)$ by $\mathcal{M}$ for $(\Phi, \Phi') = (F, F)$. Line operators do not distinguish generations. What depends on N_g is the baryon-number violation, which on every G_{SM} bundle is $\Delta B = N_g(Q_{(2)} - 18Q_{(1)})$, with $Q_{(2)}$ and $Q_{(1)}$ the $SU(2)$ and $U(1)$ topological charges in the normalization of Ref. [58].

The flavor group does depend on N_g . With generic Yukawa couplings and generic CKM matrix, and without ν_R , the invertible flavor symmetry of one sector, modulo the center of G_{SM} , is the connected group $U(1)^3$ generated by

$$X_0 = B - L, \quad X_1 = L_1 - L_2, \quad X_2 = L_2 - L_3, \quad (59)$$

compared with $U(1)_{B-L}$ for $N_g = 1$. Its cubic and gravitational 't Hooft anomalies are nonzero (e.g. $A(X_0^3) = -3$), and all 2-group coefficients $\kappa_{ab} = \sum y q_a q_b$ vanish. On every G_{SM} bundle $\Delta B = N_g \Delta L_i$, with $\gcd \Delta B = 3$

and $\gcd \Delta L_i = 1$. These data are invariant under permutations of the families, and those of the mirror sector are their negatives. Anomaly matching between the sectors then allows, among all identifications $X_a \mapsto \sum_b V_{ab} X'_b$ with $V \in GL(3, \mathbb{Z})$ and entries in $\{0, \pm 1\}$, exactly

$$\mathcal{M}_P : X \mapsto \sigma(X)', \quad \mathcal{M} : X \mapsto -\sigma(X)', \quad (60)$$

with σ one of the six permutations of the families; this extends Eq. (58). All of them preserve the global form of the flavor group. Neither the line map nor anomaly matching fixes which mirror family an ordinary family goes to; the action of $\mathcal{M}_P$ on the Yukawa couplings, which could fix σ , is not studied here. Since all κ_{ab} vanish in both sectors, the invertible symmetry is a direct product of $U(1)^3 \times U(1)^3$ with the 1-form symmetries of $\mathcal{T}_{\text{el}}$ or $\mathcal{T}_D$, and the map (57) is unchanged.

D. Non-invertible symmetries

A flavor rotation with an ABJ anomaly can survive as a non-invertible symmetry, the rotation dressed with a topological field theory [62, 63]. We first determine when this happens in one sector, for one generation. Let U_ϕ be a flavor rotation with phase $e^{2\pi i \phi_f}$ on each field f , whose instanton indices I_3, I_2 for $SU(3)$ and $SU(2)$ are integers. Its anomaly on every G_{SM} bundle is $\exp(2\pi i r P(\ell))$, where $P(\ell) = \frac{1}{2} \int \ell \cup \ell \in \mathbb{Z}$ on spin manifolds, ℓ is the class of the $U(1)_m^{(1)}$ background, and

$$r = \frac{2}{3}b_3 + \frac{1}{2}b_2 + \frac{1}{6}\hat{b} \pmod{1} \quad (61)$$

for the induced shift $\Delta(\theta_3, \theta_2, \hat{\theta}_Y) = 2\pi(b_3, b_2, \hat{b})$. Moreover $r \in \mathbb{Z}$ if and only if the shift lies in Λ_θ . For $r \in \mathbb{Z}$ the rotation is invertible; for $r = p/N$ it gives the non-invertible defect $U_\phi \otimes \mathcal{A}^{N,p}[\ell]$ of Ref. [62]; if I_3 or I_2 is not an integer there is no topological defect. The defect maps an 't Hooft line $H(m)$ to a line of fractional hypercharge $Y = r\tilde{Y}(m)$ attached to a $\mathbb{Z}_N$ surface, which is genuine if and only if $N \mid \tilde{Y}(m)$; Wilson lines are unaffected.

With all Yukawa couplings on there is no such defect: r is trivial on the whole flavor group allowed by instantons, so B , L and $B \pm L$ give only invertible elements. $U(1)_{B-L}$ is invertible but not a gauge symmetry, and the elements of B , L and $B + L$ allowed by instantons are gauge transformations or $(-1)^F$ times gauge transformations; for example, $e^{i\pi(B+L)}$ is $(-1)^F$ times a gauge transformation. Non-invertible defects appear in the limits where some Yukawa couplings vanish. For $y_e = 0$, or with both quark Yukawa couplings off, there is a $U(1)$ with only a Y^2 ABJ anomaly. Writing $r = aR$ for a rotation by $2\pi a$, a rational rotation is invertible if and only if $aR \in \mathbb{Z}$, and otherwise non-invertible with N the denominator of aR ; examples are the e_R number N_e ($R = -1$) and $X_{CK} = u^c - d^c + e^c$ of Ref. [57] ($R = 2$). The global form enters through $R(G_k) = (6/k)^2 R(G_{\text{SM}})$: the $\mathbb{Z}_{36}$ of N_e that is invertible in the cover becomes non-invertible in G_{SM} , apart from the identity.

In two sectors these defects combine with the mirror map and with the gauging of finite subgroups of the 1-form symmetry, which can produce duality defects [64–66]. The mirror sector has $R' = -R$, the anomaly is $rP(\ell) + r'P(\ell')$ with no contribution of θ_{mix} or kinetic mixing, and the defects are products; with all Yukawa couplings on there are none in either sector. In the frame D , $\mathcal{M}_P = \text{swap}(P \otimes P)$ maps $U_\phi \otimes \mathcal{A}^{N,p}[\ell] \mapsto U'_\phi \otimes \mathcal{A}^{N,-p}[\ell']$, an involution that preserves N , the fusion and the invertible subgroups, while $\mathcal{M} = \text{swap}(C \otimes 1)$ gives $U'_\phi \otimes \mathcal{A}^{N,p}$ and sends mirror defects to content that is not of Standard Model type. In the original frame $H(m)$ goes to a mirror Wilson line with $y' = \pm \tilde{Y}(m)$ (class A/B of Φ), which is non-genuine if and only if $N \nmid y'$; the image $H'(0, 0, 0, \pm 6)$ of the electron line stays genuine if and only if $N \mid 6$.

The 1-form symmetry of $\mathcal{T}_D$ is $U(1)_m \times U(1)'_e$ ($U(1)_m \times U(1)'_m$ in the frame D). Gauging $\mathbb{Z}_N \subset U(1)_m$ with discrete torsion k [54] adds the line $(k m_1, \ell_Y/N)$ and gives the line classes $\mathbb{Z} \oplus \mathbb{Z}_{\text{gcd}(k,N)}$. For $\text{gcd}(k, N) > 1$, including the standard gauging $k = 0$, torsion appears, so the gauged theory is not isomorphic to the original one and there is no defect. For $\text{gcd}(k, N) = 1$ the matter is fixed and the gauging is a shift of the coupling,

$$\mathcal{T}/\Gamma_k \cong \mathcal{T}(\tau + B_k), \quad B_k = \frac{k^{-1}}{N} \ell_Y \otimes \ell_Y, \quad (62)$$

that is, a fractional shift $\hat{\Delta}\theta_Y = 2\pi \cdot 6k^{-1}/N$, which is never a θ period. Let $\sigma_{(k,k)}$ gauge $\mathbb{Z}_N \times \mathbb{Z}_N \subset U(1)_m \times U(1)'_e$ with torsion (k, k) , $\text{gcd}(k, N) = 1$. Then $\mathcal{M}_P \circ \sigma_{(k,k)}$ maps $\mathcal{T}_D$ to itself on the locus

$$\begin{aligned} \tau'_3 &= -\bar{\tau}_3, & \tau'_2 &= -\bar{\tau}_2, & \text{Re } \hat{\tau}_{\text{mix}} &= 0, \\ \hat{\tau}'_Y &= -\bar{\hat{\tau}}_Y - \frac{6k^{-1}}{N}, \end{aligned} \quad (63)$$

where all couplings are those of the frame D ; in the original-frame mirror couplings, for $\Phi' = F$, the locus reads $\tau'_3 = 1/\bar{\tau}_3$, $\tau'_2 = 1/\bar{\tau}_2$ and $\hat{\tau}'_Y = 1/(\bar{\hat{\tau}}_Y + 6k^{-1}/N)$. On this locus $\mathcal{T}_D$ has a non-invertible self-duality defect of the type of Ref. [64], orientation reversing like $\mathcal{M}_P$, for each of the 288 elements of the class of $\mathcal{M}_P$. Since k^{-1} is defined modulo N , B_k is defined only up to $\ell_Y \otimes \ell_Y$, which is a θ period of the frame- D couplings, and the loci are understood up to such periods. We have not constructed the topological field theory of the defect or its fusion rules; its non-invertibility follows from the non-trivial gauging [64–66]. There are no such defects for $\mathcal{T}_{\text{el}}$, from $\mathcal{M}$ or $\mathcal{M}_P$ composed with any finite gauging, because the images carry magnetic matter and gauging does not move matter; nor for $\mathcal{T}_D$ from $\mathcal{M}$, whose frame form $\text{swap}(C \otimes 1)$ does not preserve the chiral content. The S-free mirror parity $\text{swap}(P \otimes P)$ gives $\mathcal{T}_{\text{el}}$ the analogous defect on the same loci.

In summary, the line operators make the slogan “ordinary electric $\leftrightarrow$ mirror magnetic” precise: the mirror maps send the heavy charges of one sector to heavy

monopoles of the other, the electron to the minimal hypercharge monopole, and they can be symmetries of the theory with matter only if the mirror matter is magnetic in the mirror’s electric frame, i.e. electric in the frame D ; for that theory the 1-form symmetries of the two sectors are exchanged with matching anomalies.

V. COUPLINGS, PORTALS AND PHENOMENOLOGY

Mirror-matter models usually take the two sectors to have equal gauge couplings, $g' = g$, as an exact mirror parity requires [2, 4, 5], and oscillations of neutrons into mirror neutrons are studied both for a degenerate and for a slightly split spectrum [6, 9, 23, 67]. The fixed loci of Sec. IIIB instead give $g' = 4\pi/g$ at $\theta = 0$, so that the mirror sector is strongly coupled in its electric frame. This section asks what that means for observations, and in particular for n – n' oscillations. There are three ways to read the relation. In option A, $\mathcal{M}_P$ is exact and the mirror sector is weakly coupled in its S-dual, magnetic frame, which reproduces our physics exactly; this is the literal content of “ordinary electric $\leftrightarrow$ mirror magnetic” of Sec. IV. In option B1, the relation is imposed only at an ultraviolet scale M_* on a theory whose matter is electric in both sectors, so that the mirror sector really is strongly coupled. In option B2, the theory above M_* is S-dual, as in option A, and the duality is broken below M_* by running and thresholds. The options have different observable consequences, and we work out all three. Since $\mathcal{M}_P$ reverses the orientation of spacetime as well as exchanging the sectors, we take it as the primary mirror map and ask what it implies for the operators that couple the two sectors, the portals: which of them $\mathcal{M}_P$ forbids, which it constrains, and whether a forbidden one is regenerated by radiative corrections (Sec. VC); the other mirror elements enter only for comparison. Where $\mathcal{M}_P$ forbids an operator, its vanishing is a result; where it does not, a vanishing coefficient, such as that of the Higgs portal $\lambda|H|^2|H'|^2$, is an assumption, which we state where it is made. For n – n' oscillations we use the parametrization of Ref. [9] by a mass splitting and a mixing angle (Sec. VB). For hypercharge kinetic mixing, which S exchanges with the mixed θ -angle and whose status therefore depends on the frame, we consider two readings: (i) ϵ is an allowed entry of the coupling matrix on which the dualities act, and (ii) ϵ is a portal and vanishes in the frame where the light matter is electric.

A. Couplings in the S-dual frame

We first translate the fixed locus into the frame D , where the mirror sector of options A and B2 is weakly coupled and looks conventional. At $\theta = 0$ the fixed locus of $\mathcal{M}_P$, Eq. (27), reads in terms of $\alpha_k = g_k^2/4\pi$

$$\alpha_3 \alpha'_3 = \alpha_2 \alpha'_2 = 1, \quad \alpha_Y \alpha'_Y = 9, \quad (64)$$

or $\alpha_1\alpha'_1 = 25$ in grand-unified normalization. In the S-dual frame $D = 1 \otimes S$ of the mirror sector, $D\mathcal{M}_PD^{-1} = \text{swap}(P \otimes P)$ on all couplings [Eq. (24)], and $\text{Fix}(\mathcal{M}_P)$ maps onto the exact mirror-parity locus

$$\tau'_k{}^D = -\bar{\tau}_k, \quad \hat{\tau}_{22}^D = -\overline{\hat{\tau}_{11}^D}, \quad \text{Re } \hat{\tau}_{\text{mix}}^D = 0, \quad (65)$$

i.e. $g'_D = g$ and $\theta'_D = -\theta$, where $\hat{\tau}^D$ is the hypercharge block (39) in the frame D ; the kinetic mixing ϵ_D in this frame is free. On the slice $\theta_Y = 0$ of $\text{Fix}(\mathcal{M}_P)$, where $\epsilon = 0$ and $g_Y g'_Y = 12\pi/\sqrt{1 - 9\theta_{\text{mix}}^2/4\pi^2}$, one finds $g_{Y,D} = g'_{Y,D} = 12\pi/g'_Y$, all $\theta_D = 0$, and

$$\epsilon_D = \pm \frac{3\theta_{\text{mix}}}{2\pi}, \quad (66)$$

with the sign set by the lift of S. The topological mixing of the original frame becomes kinetic mixing in the frame D , and $|\epsilon_D| < 1$ if and only if $|\theta_{\text{mix}}| < 2\pi/3$. Equal couplings in the electric frames are possible only at strong coupling: on $\text{Fix}(\mathcal{M}_P)$, $g = g'$ with $\theta' = \theta$ requires $|\tau| = 1$, i.e. $\alpha \geq 1$ for $SU(3)$ and $SU(2)$, and with $\theta' = -\theta$ only $\tau = i$; for the hypercharge, $\alpha_Y = 3\sqrt{1 + \epsilon^2}/\sqrt{1 - 9(\theta_Y^2 + \theta_{\text{mix}}^2)/4\pi^2} \geq 3$.

The relations (64) hold at one scale. With $d\alpha^{-1}/d\ln\mu = b/2\pi$ and the one-loop coefficients $b = (7, \frac{19}{6}, -\frac{41}{6})$ of the Standard Model for $SU(3)$, $SU(2)$ and $U(1)_Y$, running in the electric frames of both sectors gives

$$\frac{d\ln(\alpha\alpha')}{d\ln\mu} = -\frac{b}{2\pi}(\alpha + \alpha'), \quad (67)$$

which vanishes only for $b = 0$. With matter electric in both sectors, $\text{Fix}(\mathcal{M}_P)$ can therefore hold only at one scale. In the frame D the mirror content, the parity image of the ordinary one, has $b' = b$, so $g'_D(\mu) = g(\mu)$ at all scales once it holds at one. The implied electric-frame mirror coupling $\alpha' = c/\alpha$, with $c = 1, 1, 9$, then runs as $d\alpha'/d\ln\mu = cb/2\pi$, never as the $-b\alpha'^2/2\pi$ of an electric mirror Standard Model. Without states charged under both hypercharges, neither $\text{Im } \hat{\tau}_{\text{mix}}$ nor θ_{mix} runs at one loop in a frame in which all matter is electric (the frame D in option A); in option A the original-frame θ_{mix} , equal to $(2\pi/3)\epsilon_D$ on $\theta_Y = 0$ [Eq. (77)], then runs with $g_{Y,D}$.

B. Phenomenological options

We describe n - n' oscillations as in Ref. [9], by the splitting Δ of the two mass eigenvalues and a mixing angle $\theta_{nn'}$. In the flavor basis (n, n') the nonrelativistic Hamiltonian is

$$H = \bar{m} + \begin{pmatrix} -\frac{1}{2}\Delta \cos 2\theta_{nn'} & \delta_m \\ \delta_m & \frac{1}{2}\Delta \cos 2\theta_{nn'} \end{pmatrix}, \quad (68)$$

$$\delta_m = \frac{1}{2}\Delta \sin 2\theta_{nn'},$$

with $\bar{m}$ the mean mass, so that the flavor-basis splitting is $\Delta m_n \equiv m_{n'} - m_n = \Delta \cos 2\theta_{nn'}$ and the mixing mass δ_m is the off-diagonal entry. The degenerate

regime, $\Delta m_n \simeq 0$ and $\theta_{nn'} \simeq \pi/4$, and the nondegenerate one, $|\Delta m_n| \gg |\delta_m|$ and $\theta_{nn'} \ll 1$, are regions of this one parametrization. Searches with stored ultracold neutrons, with and without a mirror magnetic field [68–73], bound the oscillation time in the degenerate regime; the strongest is $\tau_{nn'} \geq 448$ s (90% C.L.) for a mirror field below 100 nT [70]. These limits assume $\Delta \cos 2\theta_{nn'} = 0$ and bound $\delta_m = \hbar/\tau_{nn'}$. They apply as long as the splitting does not dephase the oscillation between two wall collisions, $|\Delta m_n| \ll \hbar/t_f \simeq (0.7\text{--}1.3) \times 10^{-23}$ GeV, with $t_f \approx 0.05\text{--}0.1$ s the mean free-flight time of the stored neutrons [68, 69, 73]. A beam search gives $\tau_{nn'} > 1$ s for splittings between 2 and 69 peV [74]. References [67, 75] use the mixing-angle framework of Eq. (68) in a nondegenerate region, with Δm_n up to about 1.2 μeV and magnetic fields up to 6.6 T, and the regeneration search of Ref. [75] excludes part of the region proposed in Ref. [67]. The region of Ref. [9], $\Delta \simeq 3\text{--}12$ μeV with $\sin^2 2\theta_{nn'} \simeq 0.8\text{--}2 \times 10^{-5}$, so that $\Delta m_n \simeq \Delta$, lies above both: the degenerate-case limits do not apply to it, and the searches of Refs. [67, 75] do not reach it. For orientation, the rect-angle of these ranges corresponds to $\delta_m \approx 4.2 \times 10^{-18}\text{--}2.7 \times 10^{-17}$ GeV, with $\delta_m \simeq 1.16 \times 10^{-17}$ GeV at the typical point $\Delta = 6.578$ μeV , $\sin^2 2\theta_{nn'} = 1.25 \times 10^{-5}$, of Ref. [8]. We evaluate the options with one-loop running and the particle data of Ref. [76]. The Standard Model reaches the self-dual values of Sec. IIIB only far from the electroweak scale (Fig. 1): $\alpha_3 = 1$ at $\mu_3 \simeq 0.287$ GeV, which is formal since QCD is nonperturbative there; $\alpha_2 = 1$ at $\mu_2 \simeq 2.15 \times 10^{-23}$ GeV, also formal since $SU(2)$ is Higgsed; and $\alpha_Y = 3$ at $\mu_Y \simeq 1.26 \times 10^{41}$ GeV, above the Planck mass $M_P = G^{-1/2}$, with the Landau pole at 1.71×10^{41} GeV. Varying α_s , $\hat{\alpha}^{-1}(M_Z)$ and $\sin^2 \hat{\theta}_W(M_Z)$ by $\pm 1\%$ and the quark thresholds by factors of 2, jointly, gives $\log_{10}(\mu_3/\text{GeV}) \in [-0.61, -0.47]$ and $\log_{10}(\mu_Y/\text{GeV}) \geq 40.59$. At two to four loops in the QCD β function [77, 78], with μ_3 defined by $\alpha_s = 1$ in the theory with three light flavors, μ_3 rises to 0.527–0.646 GeV, and since at $\alpha_s = 1$ the four-loop term of the β series exceeds the three-loop one, μ_3 is determined only in order of magnitude. Still, $\mu_3 < 1$ GeV at every loop order and for every input variation, and $\mu_Y > M_P$ throughout, so no scale between about 1 GeV and M_P is self-dual for any factor. Elsewhere we quote one-loop values.

The options are summarized in Table V. In option A, $\mathcal{M}_P$ is exact and the mirror matter is electric in the frame D : $g'_D(\mu) = g(\mu)$, $\theta'_D = -\theta$ and $\theta_{\text{mix},D} = 0$ at every scale. The electric-frame mirror couplings at M_Z are $\alpha'_3 = 8.47$, $\alpha'_2 = 29.6$ and $\alpha'_Y = 885$, and they stay above 1 between 2 GeV and M_P . The spectrum is an exact parity image, $m_{n'} = m_n$, with the consequences for n - n' mixing given below. This requires that per-sector S be an exact equivalence of the chiral Standard Model, which is not established (App. B), and that $\mathcal{M}_P$ also hold for the Yukawa couplings and the Higgs sector. As usual in mirror cosmology, the symmetry is exact in the Lagrangian while the initial conditions break it: asymmetric reheat-

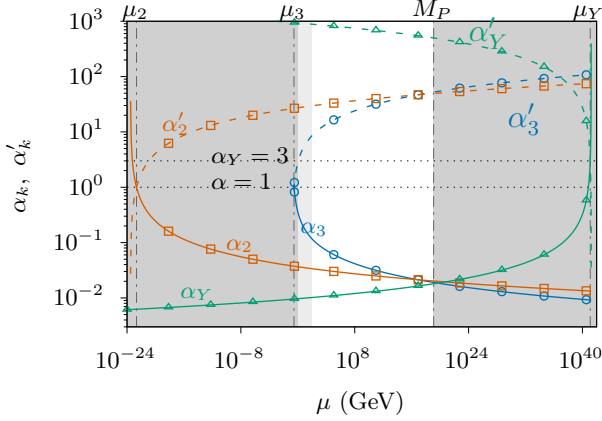


FIG. 1. One-loop running of the Standard Model couplings $\alpha_k = g_k^2/4\pi$ (solid) and of the mirror couplings $\alpha'_k = c_k/\alpha_k$, $c = (1, 1, 9)$ (dashed), on the fixed locus of $\mathcal{M}_P$ in the electric frame, for $k = 3$ (blue circles), 2 (vermilion squares) and Y (green triangles), with the particle data of Ref. [76] and α_Y in the normalization $Q = T_3 + Y$; the curves are also labeled directly. Dotted lines mark the self-dual values $\alpha_3 = \alpha_2 = 1$ and $\alpha_Y = 3$. Vertical lines mark $\mu_2 \simeq 2.1 \times 10^{-23}$ GeV ($\alpha_2 = 1$), $\mu_3 \simeq 0.29$ GeV ($\alpha_3 = 1$), $M_P = G^{-1/2}$ and $\mu_Y \simeq 1.3 \times 10^{41}$ GeV ($\alpha_Y = 3$). Gray shading marks where the Standard Model is not valid, so that the curves are formal one-loop extrapolations: $\mu > M_P$ and $\mu < 1$ GeV (darker gray; below 1 GeV for α_3 , the QCD regime), and $\mu < M_Z$ (the whole shaded region on the left, for α_2 and α_Y). The curves α_3 and α'_3 are drawn only above 0.25 GeV, and the vertical axis is cut at 10^3 .

ing leaves the mirror sector colder, $T' < T$ [4], which the cosmological bounds of Sec. V D assume. The degeneracy $m_{n'} = m_n$ does not depend on T'/T . In option B1, $\text{Fix}(\mathcal{M}_P)$ holds only at a point M_* , with matter electric in the electric frames of both sectors, so that $\alpha'_k(M_*) = c_k/\alpha_k(M_*)$. With this matter $\mathcal{M}_P$ maps the theory to one with dynamical monopoles in both sectors and is not a symmetry (Sec. IV B), not even at M_* . Option B1 is therefore a bare boundary condition, $\text{Fix}(\mathcal{M}_P)$ imposed on the gauge couplings alone, with no symmetry behind it; we keep it because it is the only option in which the coupling relation has an observable consequence. For every $M_* \in [10^3 \text{ GeV}, M_P]$ this gives $\alpha'_3 \geq 11$, $\alpha'_2 \geq 30$ and $\alpha'_Y \geq 499$ at M_* ; mirror color confines at a formal one-loop scale $\Lambda'_3 \gtrsim 0.92 M_*$, so $m_{n'}/m_n \sim M_*/\text{GeV}$, Δm_n is of order M_* , and $\sin^2 2\theta_{nn'} \simeq 4\delta_m^2/\Delta m_n^2$ is suppressed by the square of δ_m/M_* . A degenerate n' would need $\alpha_3(M_*)$ to equal 1 almost exactly, which happens only near μ_3 , not at an ultraviolet scale. In option B2, the theory above M_* is S-dual and the mirror matter is electric in the frame D , while the mirror duality is broken below M_* by thresholds or supersymmetry breaking. Like option A, this needs an S-dual completion whose low-energy matter is chiral, which is not established: App. B finds S exact only in the $\mathcal{N} = 4$ theory, which has no

chiral matter. A mismatch δ_3 of α_3^{-1} between the sectors splits the neutron masses by

$$\Delta m_n \simeq \frac{2\pi}{9} \delta_3 m_n. \quad (69)$$

A mirror neutron within a few μeV of the neutron therefore needs an extreme tuning of δ_3 (quantified below for $v' \neq v$), while a 1% splitting of one heavy Dirac color triplet gives $\delta_3 \simeq 1.1 \times 10^{-3}$ and $\Delta m_n \simeq 0.69$ MeV. Since ϵ_D does not enter α_3 , these numbers hold in both readings of kinetic mixing. Below M_* , option B2 is conventional mirror matter whose exchange symmetry, $g'_D = g$ at M_* , is broken softly by thresholds, so its tuning requirements, here and below, hold for any such model and are not consequences of the self-duality; they belong to the frame-dependent content of Sec. III H.

a. n - n' with $\mathcal{M}_P$ exact. In the frame D , $\mathcal{M}_P$ acts as swap ($P \otimes P$), which is the standard mirror parity. Its consequences for n and n' are therefore those of any model with exact mirror parity, equal masses [4, 5, 23] and maximal mixing for any nonzero mixing mass [23]; what we add is their form in the parametrization (68) and the phases that the lift of $\mathcal{M}_P$ fixes. In option A, $\mathcal{M}_P$ constrains the n - n' mass matrix; in option B2 it constrains only the six-quark coefficients at M_* (Sec. V C). With gravity, these statements are exact only if $\mathcal{M}_P$ is gauged (Sec. V D). We write the most general Lorentz-scalar mass term of $\Psi = (n, n')$ as $-\bar{\Psi}(S + i\mathbf{P}\gamma^5)\Psi$, with S and $\mathbf{P}$ Hermitian (eight real parameters). With the lift of $\mathcal{M}_P$ to fermions that has $\eta_P = +1$, four survive:

$$S = \begin{pmatrix} m & a \\ a & m \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} m_5 & ib \\ -ib & -m_5 \end{pmatrix}, \quad (70)$$

with m , a , m_5 and b real. The flavor-basis diagonal entries are equal, $m_{n'} = m_n$, the γ^5 masses are opposite, as $\theta' = -\theta$ in Sec. III H, and the masses are $R \mp a$, $R = (m^2 + m_5^2 + b^2)^{1/2}$. Hence $\Delta \cos 2\theta_{nn'} = 0$ exactly, and in the Hamiltonian (68), the rest-frame Dirac Hamiltonian on its positive-energy subspace in the flavor states $|n\rangle$ and $|n'\rangle = \mathcal{M}_P|n\rangle$,

$$\theta_{nn'} = \frac{\pi}{4}, \quad \Delta = 2|\delta_m|, \quad \delta_m = a, \quad (71)$$

for every b and every $\delta_m \neq 0$ with $|a| < R$. Every mass eigenstate is an eigenstate of $\mathcal{M}_P$ with equal n and n' content, at $b = 0$ the states $(n \pm n')/\sqrt{2}$; for $b \neq 0$ the mixing angles of the Weyl components differ from $\pi/4$, but Eq. (71) is unchanged. These statements hold for every lift of $\mathcal{M}_P$ and in both frames D and D' ; only the phases depend on the lift: with $\mathcal{M}_P \circ (-1)^{F'}$ the entry S_{12} is imaginary and P_{12} real. The phase condition for $\eta_P = +1$ is also the one imposed by $CP \otimes CP$ in each sector.

Exact $\mathcal{M}_P$ is therefore the degenerate regime of the storage experiments, and the region of Ref. [9] is out of its reach, for four reasons taken together. Exact $\mathcal{M}_P$ forces $\Delta \cos 2\theta_{nn'} = 0$. In that regime $\tau_{nn'} \geq 448$ s (90%

C.L.) [70] gives $\delta_m \leq 1.5 \times 10^{-27}$ GeV. The values of δ_m in the region of Ref. [9] are larger by a factor $2.9 \times 10^9 - 1.8 \times 10^{10}$. They escape the storage limits only if $|\Delta m_n| \gg \hbar/t_f$, a flavor-diagonal splitting that exact $\mathcal{M}_P$ forbids. The region of Ref. [9] therefore requires $\mathcal{M}_P$ to be broken in the flavor-diagonal entries.

b. Soft breaking by $v' \neq v$. In the mirror-matter model that motivates this work the mirror symmetry is broken by $(v' - v)/v = \varepsilon$ (Sec. I). If every other coupling is mirror symmetric, all mirror quark masses scale as $(1 + \varepsilon)m_q$, and

$$\Delta m_n = \kappa m_n \varepsilon, \quad \kappa = \frac{\sigma_N}{m_n} + h \left(1 - \frac{\sigma_N}{m_n}\right), \quad (72)$$

where $\sigma_N = \sigma_{\pi N} + \sigma_s$ is the light-quark sigma term and $h = d \ln \Lambda_3 / d\varepsilon$ the effect of the heavy-quark thresholds on the QCD scale: $h = 2/9$ at one loop, the leading-order heavy-quark relation of Ref. [79], and $0.197 \leq h \leq 0.287$ over one to four loops and two threshold prescriptions. With $\sigma_{\pi N} = 50 \pm 10$ MeV, a band that brackets the Roy–Steiner [80] and lattice [81] determinations, and $\sigma_s = 45 \pm 25$ MeV, which brackets the lattice averages of Ref. [81], we find $\kappa \simeq 0.30$ (range 0.25–0.39): $v' \neq v$ moves the QCD scale only through the heavy-quark thresholds, so most of the nucleon mass does not scale with v . In the parametrization (68) this is the flavor-basis splitting, $\Delta \cos 2\theta_{nn'} = \kappa m_n \varepsilon$, so that $\tan 2\theta_{nn'} = 2\delta_m / (\kappa m_n \varepsilon)$; it is 0.28 – 2.8 μeV for $\varepsilon = 10^{-15}$ – 10^{-14} at central κ , and 0.23 – 3.6 μeV over the κ band. The window $\Delta \simeq 3$ – 12 μeV of Ref. [9], where $\Delta \cos 2\theta_{nn'} \simeq \Delta$, then needs $\varepsilon \simeq 1.1$ – 4.2×10^{-14} at central κ , at and above the upper end of the rough range $\varepsilon = 10^{-15}$ – 10^{-14} . This assumes that $v' \neq v$ is the only breaking. References [8, 9] assume instead a universal relative splitting of all masses, $\Delta/m \simeq 7 \times 10^{-15}$ in Ref. [8], which is the case $\kappa = 1$; with $\kappa = 1$ the window needs $\varepsilon \simeq 3.2 \times 10^{-15}$ – 1.3×10^{-14} , inside the rough range, and the typical point above needs $\varepsilon \simeq 7.0 \times 10^{-15}$. Where the window lies relative to the rough range thus depends on how the breaking reaches the QCD scale. These relations assume that the off-diagonal entry keeps the phase fixed by $\mathcal{M}_P$, equivalently by $CP \otimes CP$, which $v' \neq v$ does not break at $\theta = \theta' = 0$, and they reduce to Eq. (71) at $\varepsilon \rightarrow 0$. The window itself is the value of Ref. [9], fitted to neutron data and cosmology, and not a result of this paper. For option B2 the Higgs-vev asymmetry sets the benchmark for the tuning. The mismatch that gives the same splitting as ε is $\delta_3 = 9\kappa\varepsilon/2\pi \simeq 4.3 \times 10^{-16}$ – 4.3×10^{-15} , so the gauge mismatch must satisfy $|\delta_3| \lesssim 4 \times 10^{-16}$ – 4×10^{-15} in order not to dominate the splitting, which is still an extreme tuning.

c. The mixing angle. We treat $\theta_{nn'}$, equivalently δ_m , as a phenomenological input. In the setting of this paper, one generation per sector, no right-handed neutrinos and no portal operators, δ_m , which changes B and B' by one unit each ($\Delta B = -\Delta B' = 1$ with mirror content), has no source in any topological sector of any of the 30 global forms $(\tilde{G} \times \tilde{G}')/\Gamma$, $\Gamma \subset \mathbb{Z}_6 \times \mathbb{Z}_6'$,

on closed spin four-manifolds, with gauge instantons and closed spin gravitational backgrounds such as K3: on every such background $\Delta(B - L)$ is even, while δ_m needs $\Delta(B - L) = 1$. This holds with or without $\mathcal{M}_P$: $\mathcal{M}_P$ preserves the lattice of allowed violations of (B, L, B', L') and keeps the n – n' transition outside it, so it neither supplies nor removes a source. The doubled transition $nn \leftrightarrow n'n'$ is in the lattice, but it is not a mass mixing. Gravitational instantons with a nontrivial boundary, wormholes and black holes are not covered, nor are the non-orientable backgrounds that a gauged $\mathcal{M}_P$ would bring in (Sec. VD). A nonzero $\theta_{nn'}$ therefore needs explicit breaking of $B - L$ and $B' - L'$ between the sectors, such as a six-quark portal (Sec. VC), or effects outside this scope; we do not commit to a mechanism. For scale only: a $B - L$ -violating six-quark operator $(udd)(u'd'd')/M_P^5$ gives $\delta_m \sim 2 \times 10^{-100}$ – 3×10^{-99} GeV, about 82 to 83 orders of magnitude below the typical value above. This is a dimensional estimate, as Ref. [23] makes for the same operators with a lower scale, with a coefficient of order one and a hadronic matrix element Λ_h^6 , $\Lambda_h = 0.2$ – 0.3 GeV. Since the sectors share no gauge interaction, the matrix element factorizes into one three-quark matrix element per sector, $\beta\beta'^*$ with $|\beta'| = |\beta|$, so Λ_h^6 stands for $|\beta|^2$; the range of Λ_h brackets the cube root, $\simeq 0.24$ GeV, of the lattice proton-decay matrix element $\beta = 0.0144(15)$ GeV³ [Eq. (23) of Ref. [82]].

Most earlier studies of δ_m look for a perturbative origin of n – n' oscillations, in six-quark operators and similar portals between the sectors [4, 23]. Studies such as Refs. [8, 9], which describe the oscillations by a mixing angle, as for neutrino oscillations, may be better understood as describing a nonperturbative, algebraic mismatch of the vacua of the two sectors. The oscillation parameters may then have a topological origin in new physics or in dimensions that this work does not explore.

C. Portal operators under $\mathcal{M}_P$

Beyond gravity, the two sectors communicate through portal operators built from fields of both sectors, which control links between them such as millicharges and n – n' mixing. We therefore ask which portals $\mathcal{M}_P$ forbids, which it constrains, and which of the forbidden ones stay zero. The mirror elements act classically only in an S-dual frame in which all matter is electric, and we classify the portals there. In the frame D , with mirror content, $\mathcal{M}_P$ is swap $(P \otimes P)$. In the frame D' , with copy content, it is the conjugate of this form by the relabeling $U = 1 \otimes C$, which on fermions is swap $(CP \otimes PC)$, with PC denoting C followed by P ; it squares to 1 on fermions. The lattice label swap $(CP \otimes CP)$ of Table II, read on fermions with the C and P of App. A, is instead $\mathcal{M}_P \circ (-1)^{F'}$; the two differ only on operators with an odd number of fermions in a sector. These forms are physical only if per-sector S is exact (option A, or above M_* in option B2); in the original frame of option B1 there is

TABLE V. Phenomenological options for $\mathcal{M}_P$ (one loop, particle data of Ref. [76]). “Matter electric in” is the frame in which the mirror matter is electric; the couplings are those of the electric frame of the mirror sector; the last column gives the n' mass and the n - n' mixing in the parametrization (68), with $\theta_{nn'}$ and δ_m phenomenological inputs. In option A the mixing is that of the exact $\mathcal{M}_P$, Eq. (71); a Higgs-vev asymmetry $(v' - v)/v = \varepsilon$ breaks it softly and adds $\Delta \cos 2\theta_{nn'} = \kappa m_n \varepsilon$ [Eq. (72)], which the region of Ref. [9] requires. The requirements on δ_3 in option B2 are discussed in the text. The value of α'_Y in option A is that at $\epsilon_D = 0$; in reading (i) it is $885.15/(1 - \epsilon_D^2)$. Options A and B2 require per-sector S to be exact for chiral matter, which is not established (App. B); in option B1 $\mathcal{M}_P$ is not a symmetry, and $\text{Fix}(\mathcal{M}_P)$ is a boundary condition on the couplings at M_* (Sec. IV B).

Option	$\text{Fix}(\mathcal{M}_P)$ holds	Matter electric in	Mirror couplings	n' and n - n' mixing
A	at all scales	frame D	$\alpha'_3 = 8.47$, $\alpha'_2 = 29.6$, $\alpha'_Y = 885$ at M_Z	$m_{n'} = m_n$; $\Delta \cos 2\theta_{nn'} = 0$, $\theta_{nn'} = \pi/4$, $\Delta = 2 \delta_m $
B1	imposed at M_*	both electric frames	$\alpha'_3 \geq 11$, $\alpha'_2 \geq 30$, $\alpha'_Y \geq 499$ at M_*	$m_{n'}/m_n \sim M_*/\text{GeV}$; $\sin^2 2\theta_{nn'} \simeq 4\delta_m^2/\Delta m_n^2$
B2	at M_* only	frame D	$g'_D = g$ at M_* , broken below	$\Delta m_n \simeq (2\pi/9)\delta_3 m_n$ from the gauge mismatch

no selection rule on the matter portals. Since $\mathcal{M}_P$ is the standard mirror parity in the frame D , its selection rules hold in any model with exact mirror parity. Studies of such models list the renormalizable couplings between the sectors, the Higgs portal and kinetic mixing [4, 5, 83], and the neutrino and six-quark operators [4, 23], but do not classify the portals by mirror parity. What we add is that classification, complete up to dimension 6 and with the number of surviving parameters, its dependence on the lift of $\mathcal{M}_P$ to fermions, the dictionary between the frames, and the comparison with $\tilde{\mathcal{M}}$ and with the formal elements.

We take all gauge-invariant portals up to dimension 6 in an on-shell basis: the renormalizable $|H|^2|H'|^2$, $B_{\mu\nu}B'^{\mu\nu}$ and $B_{\mu\nu}\tilde{B}'^{\mu\nu}$; 24 real dimension-6 bosonic couplings, among them $J_H \cdot J'_H$, with $J_H^\mu = iH^\dagger \overleftrightarrow{D}^\mu H$; 35 current-current couplings, each coupling a current of one sector, fermion or Higgs, to a fermion current of the other; 12 Yukawa-portal and 12 dipole-portal couplings; and the neutrino portals $(LH)(L'H')$ of dimension 5 and $(LH)(L'D_\mu H')$ of dimension 6, with two real parameters each. The dimension-6 portals have 85 real parameters. We add the dimension-9 six-quark operators that mix n and n' , with 10 real coefficients. They are built from the three-quark spinors $(udd) = \epsilon_{abc}(u_a^T C d_b) d_c$ and $(QQd) = \epsilon_{abc}\epsilon_{ij}(Q_{ai}^T C Q_{bj}) d_c$, with color indices a, b, c and isospin indices i, j , and their mirror images $(u'd'd')$ and $(Q'Q'd')$, with all fields primed: the four products of an ordinary and a mirror spinor, such as $(udd)(u'd'd')$, and the product $[SSS]$ of three cross-sector scalar quark bilinears. The Supplemental Material lists the operators of every family and their images under each mirror element.[84]

The result is collected in Table VI. $\mathcal{M}_P$ forbids exactly two operators: the topological mixing $\theta_{\text{mix},D} B_{\mu\nu}\tilde{B}'^{\mu\nu}$ of the frame D , as in Sec. VD, and the dimension-6 neutrino portal $(LH)(L'D_\mu H')$ with its conjugate, which $\mathcal{M}_P$ maps to minus itself in both frames. The second prohibition holds for every lift of $\mathcal{M}_P$ that squares to +1 on the lepton doublets, of which $\eta_P = +1$ is one; a lift

that squares to -1 there, such as $\mathcal{M}_P \circ (-1)^{F'}$ or $\mathcal{M}_P$ composed with the lepton-number rotation $e^{i\pi(L+L')/2}$, allows the operator. This operator also changes $B - L$ and $B' - L'$ by one unit each, so the accidental symmetries $U(1)_{B-L} \times U(1)_{B'-L'}$ of the theory without portals already forbid it; the prohibition by $\mathcal{M}_P$ adds something once lepton number is broken, and then only for such a lift. Every other portal is allowed but constrained: coefficients are tied to those of their mirror images, P-odd pairs are antisymmetric, e.g. $c(|H|^2 X' \tilde{X}') = -c(|H'|^2 X \tilde{X})$, and some phases are fixed, so that about half of the parameters survive. With mirror content every six-quark operator has $\Delta B = -\Delta B' = \pm 1$, the portal $(LH)(L'H')$ has $\Delta L = -\Delta L'$, and $(LH)(L'D_\mu H')$ has $\Delta L = \Delta L'$; with copy content the patterns are reversed. For the lift with $\eta_P = +1$ the five surviving six-quark coefficients are $c_{(udd)(u'd'd')}$, $c_{(QQd)(Q'Q'd')}$ and $c_{(udd)(u'd'd')}^{[SSS]}$, which are real, and the pair $c_{(QQd)(u'd'd')} = c_{(udd)(Q'Q'd')}^*$; for $\mathcal{M}_P \circ (-1)^{F'}$ the first three are imaginary and $c_{(QQd)(u'd'd')} = -c_{(udd)(Q'Q'd')}^*$.

For comparison, the other mirror element compatible with the chiral content, $\tilde{\mathcal{M}}$, which is swap with copy content and swap $(C \otimes C)$ with mirror content, forbids no portal: it fixes the phase of the dimension-6 neutrino portal and keeps 8 of the 10 six-quark coefficients, and otherwise has the counts of $\mathcal{M}_P$. The formal elements $\mathcal{M} \rightarrow \text{swap}(C \otimes 1)$ and $\tilde{\mathcal{M}}_P \rightarrow \text{swap}(CP \otimes P)$ forbid more on the bosonic side, $\mathcal{M}$ both ϵ and θ_{mix} and $\tilde{\mathcal{M}}_P$ the kinetic mixing ϵ_D , and both forbid $J_H \cdot J'_H$; but they are incompatible with the chiral content, so their selection rules do not extend to the fermionic portals. Among the mirror elements compatible with chiral matter, $\mathcal{M}_P$ is therefore the only one that forbids any portal.

a. *Radiative status.* A forbidden operator is exactly zero only if nothing regenerates it. The angle $\theta_{\text{mix},D}$ vanishes (0 or 2π modulo 4π) to all orders while $\mathcal{M}_P$ is exact (with gravity, only if $\mathcal{M}_P$ is gauged; Sec. VD), and no $\mathcal{M}_P$ -symmetric portal regenerates it; where $\mathcal{M}_P$ is broken it is still not renormalized (Sec. VD). The neutrino portals and the six-quark operators each change $B - L$

TABLE VI. Portal operators between the sectors under the mirror elements, in the S-dual frame where all matter is electric (frames D and D' give the same pattern, operator by operator), with one Standard Model generation per sector. Entries give the number of real parameters that survive out of the number in the first column; for the one-parameter operators, “allowed” or “forbidden”. “Phase fixed” means that one real combination survives. $\mathcal{M}$ and $\tilde{\mathcal{M}}_P$ are not symmetries of chiral matter (Table II), so they have no selection rule on the fermionic portals (n/a). The entries for $(LH)(L'D_\mu H')$ and the fixed phases assume a lift of $\mathcal{M}_P$ that squares to +1 on the lepton doublets, such as $\eta_P = +1$. The row ϵ_D refers to reading (i) of kinetic mixing; in reading (ii) $\epsilon_D = 0$ by assumption. The operators of each family, and the $\Delta S = 2$ operators, which need two generations, are listed in the Supplemental Material.

Portal (real parameters)	$\mathcal{M}_P$	$\tilde{\mathcal{M}}$	$\mathcal{M}$ (formal)	$\tilde{\mathcal{M}}_P$ (formal)
$\lambda H ^2 H' ^2$ (1)	allowed	allowed	allowed	allowed
$\epsilon_D B_{\mu\nu} B'^{\mu\nu}$ (1)	allowed	allowed	forbidden	forbidden
$\theta_{\text{mix},D} B_{\mu\nu} \tilde{B}'^{\mu\nu}$ (1)	forbidden	allowed	forbidden	allowed
dimension-6 bosonic (24)	13/24	13/24	12/24 ($J_H \cdot J'_H$ forbidden)	12/24 ($J_H \cdot J'_H$ forbidden)
current–current (35)	20/35	20/35	n/a	n/a
Yukawa portals (12)	6/12	6/12	n/a	n/a
dipole portals (12)	6/12	6/12	n/a	n/a
$(LH)(L'H')$ (2)	1/2 (phase fixed)	2/2	n/a	n/a
$(LH)(L'D_\mu H')$ (2)	0/2	1/2 (phase fixed)	n/a	n/a
six-quark, dimension 9 (10)	5/10	8/10	n/a	n/a

by one unit, and $B' - L'$ by one unit. Unlike B and L separately, which have an $SU(2)^2$ anomaly, $B - L$ has no mixed anomaly with any gauge factor, only gravitational and cubic ones, and every other portal conserves B , L , B' and L' . Set to zero together, these operators are therefore not regenerated by any perturbative or gauge-instanton effect; only gravity can generate them. One of them alone regenerates another only through electroweak instantons. The product of $(LH)(L'H')$, with $(\Delta B, \Delta L, \Delta B', \Delta L') = (0, 1, 0, -1)$, and a six-quark operator, with $(1, 0, -1, 0)$, carries $(1, 1, -1, -1)$, the charge of one $SU(2)$ instanton and one $SU(2)'$ anti-instanton in the lattice of Sec. VB; so each regenerates the other through one electroweak instanton per sector, each vertex weighted by about $e^{-2\pi/\alpha_2} \approx 2 \times 10^{-81}$. The dimension-6 neutrino portal changes $(B - L) + (B' - L')$, which is conserved in every topological sector, and is thus exactly zero while $\mathcal{M}_P$ is exact, up to gravity. The Higgs portal, which $\mathcal{M}_P$ allows, is regenerated at one loop by the kinetic mixing ϵ_D that $\mathcal{M}_P$ also allows:

$$16\pi^2 \frac{d\lambda}{d\ln\mu} = \frac{3}{4} \frac{g_Y^2 g_Y'^2 \epsilon_D^2}{(1 - \epsilon_D^2)^2}. \quad (73)$$

With $\lambda = 0$ at the Planck mass, fixed couplings $g_Y = g_Y' = 0.3575$ (the value at M_Z), the leading logarithm $\ln(M_P/M_Z) = 39.4$ and $\epsilon_D = 4.2 \times 10^{-10}$, at the cosmological bound of Sec. VD, this gives $|\lambda| \approx 5.4 \times 10^{-22}$. Gravity also connects the Higgs sectors. With nonminimal couplings $\mathcal{L} \supset +\xi R|H|^2 + \xi' R|H'|^2$, in the convention in which the Einstein–Hilbert term is $-R/16\pi G$, and the potential $V \supset m^2|H|^2 + m'^2|H'|^2 + \lambda|H|^2|H'|^2$, tree-level graviton exchange gives $\lambda = 16\pi G(\xi m'^2 + \xi' m^2)$, i.e. $\lambda \approx -2.64 \times 10^{-33}(\xi + \xi')$ for $m^2 = m'^2 = -m_h^2/2$. Loop-level gravity, gravitational $B - L$ violation, and the regeneration of the dimension-6 portals by a nonzero ϵ_D or λ are not computed.

D. When does kinetic mixing vanish exactly?

Kinetic mixing is the renormalizable portal that would give mirror particles a small ordinary electric charge, and its size is bounded by cosmology and laboratory searches. The fixed loci of Sec. IIID constrain ϵ and θ_{mix} only through the duality element. We now ask which cases force them to vanish exactly, rather than to be small. We take one chiral generation per sector and no states charged under both hypercharges.

a. Classical symmetries. In a frame in which all light matter is electric, a symmetry is classical and acts on (B, B') as a signed permutation. We label it by (s, p, a_1, a_2) , where $s = 1$ for the swap of the sectors, $p = 1$ for P and $a_i = 1$ for C on sector i . It acts on the mixing parameter as

$$\hat{\tau}_{\text{mix}} \mapsto \begin{cases} (-1)^{a_1+a_2} \hat{\tau}_{\text{mix}}, & p = 0, \\ -(-1)^{a_1+a_2} \hat{\tau}_{\text{mix}}, & p = 1, \end{cases} \quad (74)$$

so its fixed locus forces $\epsilon = 0$ if and only if $a_1 + a_2$ is odd, and forces θ_{mix} if and only if $p + a_1 + a_2$ is odd, to 0 exactly or to $\{0, 2\pi\}$ modulo 4π up to θ periods, which never relax the condition on ϵ . The elements compatible with the chiral content are 1, $CP \otimes CP$, swap($C \otimes C$) and swap($P \otimes P$) for mirror content ($R' = \bar{R}$), and 1, $CP \otimes CP$, swap and swap($CP \otimes CP$) for copy content ($R' = R$). All have $a_1 + a_2$ even: no symmetry compatible with the chiral content forces $\epsilon = 0$. The angle θ_{mix} is forced only by the elements with $p = 1$, $CP \otimes CP$, swap($P \otimes P$) and swap($CP \otimes CP$). The sign-flipped exchange swap($C \otimes 1$), i.e. $B \rightarrow -B'$, $B' \rightarrow B$, with or without P, and a separate charge conjugation of one sector, as in Ref. [85], would forbid ϵ but need a vectorlike sector, so they are excluded here. The involution $B \leftrightarrow -B'$, which is swap($C \otimes C$), is compatible with mirror content but allows ϵ . Weyl elements act trivially

on the hypercharge block, and the only sign flip of B in $\text{Aut}(G_{\text{SM}})$ is C , together with conjugation on $SU(3)$. Compatibility is tested here on the gauge representations only; it is necessary, not sufficient.

b. Duality elements. The four classes of mirror elements of Table II take classical forms on the hypercharge block in the frames D and D' . Modulo $C \otimes C$, which acts trivially on the block, $\mathcal{M} \rightarrow \text{swap}(C \otimes 1)$ forces $\hat{\tau}_{\text{mix}} = 0$; $\mathcal{M}_P \rightarrow \text{swap}(P \otimes P)$ forces $\theta_{\text{mix},D} = 0$ with ϵ_D free; $\tilde{\mathcal{M}} \rightarrow \text{swap}$ forces nothing; and $\tilde{\mathcal{M}}_P \rightarrow \text{swap}(CP \otimes P)$ forces $\epsilon_D = 0$ with $\theta_{\text{mix},D}$ free. None of them is classical in the original frame, where $\mathcal{M}_P$ and $\tilde{\mathcal{M}}_P$ force neither parameter and Eqs. (41) and (42) hold. Under a change of frame $\gamma_1 \otimes \gamma_2$, with $\gamma_i \in SL(2, \mathbb{Z})$ acting on sector i and $C = \text{diag}(c_1, c_2)$, $D = \text{diag}(d_1, d_2)$ their lower entries,

$$\hat{\tau}_{\text{mix}} \mapsto \frac{\hat{\tau}_{\text{mix}}}{\det(C\hat{\tau} + D)}, \quad (75)$$

and under $P \otimes P$, $\hat{\tau}_{\text{mix}} \rightarrow -\hat{\tau}_{\text{mix}}$. Hence $\hat{\tau}_{\text{mix}} = 0$, i.e. $\epsilon = \theta_{\text{mix}} = 0$, is invariant under per-sector changes of frame, while $\epsilon = 0$ alone or $\theta_{\text{mix}} = 0$ alone is preserved only by changes without an S part ($c_1 = c_2 = 0$). Neither is invariant under the sector-mixing elements of $Sp(4, \mathbb{Z})$ allowed by the glue (Sec. III D): for example $\hat{\tau} \mapsto U\hat{\tau}U^T$ with $U = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}$ gives $\hat{\tau}'_{\text{mix}} = 6\hat{\tau}_Y$. A per-sector self-duality $\gamma \otimes 1$, $\gamma \neq 1$, with a fixed point in the upper half-plane (γ elliptic, or $\gamma = -1$) forces $\hat{\tau}_{\text{mix}} = 0$. On the exact $\text{Fix}(\mathcal{M}_P)$, $|\theta_{\text{mix}}| < 2\pi/3$, and $\text{Fix}(\mathcal{M}_P) \cap \text{Fix}(CP \otimes CP)$ is the point $\theta_Y = \theta'_Y = \theta_{\text{mix}} = \epsilon = 0$, $g_Y g'_Y = 12\pi$. The elements that force a mixing parameter to zero in a frame with electric matter, $\mathcal{M}$ and $\tilde{\mathcal{M}}_P$, have $a_1 + a_2$ odd and are incompatible with chiral matter, as in Table II.

c. Global forms. The global form of $(\tilde{G} \times \tilde{G}')/\Gamma$, with $\tilde{G}$ the cover and $\Gamma \subset \mathbb{Z}_6 \times \mathbb{Z}'_6$, fixes the period of θ_{mix} . The θ_{mix} -term is $\theta_{\text{mix}}N/2$, with $N = \int \frac{F}{2\pi} \wedge \frac{F'}{2\pi}$, so at fixed θ_Y, θ'_Y and non-abelian angles the period of θ_{mix} on spin manifolds is $4\pi/\text{gcd } N$: 4π for $G_{\text{SM}} \times G'_{\text{SM}}$, with $\{0, 2\pi\}$ the values invariant under P and CP ; 2π for the diagonal $\mathbb{Z}_6$ quotient; $\pi/9$ for the cover; and $4\pi|\Gamma_1||\Gamma_2|/36$ for the product forms $\Gamma = \Gamma_1 \times \Gamma_2$. No connected global form forbids ϵ , since Γ is central and the field strengths are neutral; nor does any forbid states charged under both hypercharges. $G_{\text{SM}} \times G'_{\text{SM}}$ allows such states, if they are color and weak singlets, only with integer (Y, Y') , e.g. $(1, 1)$ but not $(\frac{1}{6}, \frac{1}{6})$; the diagonal $\mathbb{Z}_6$ quotient allows $(\frac{1}{6}, -\frac{1}{6})$. Gauging an outer automorphism forbids ϵ if and only if the element has $a_1 + a_2$ odd, which is excluded for chiral sectors.

d. Radiative stability. Without states charged under both hypercharges the off-diagonal one-loop coefficient b_{12} of the hypercharge block, a sum of yy' over Weyl fermions and scalars, vanishes for mirror and for copy content, so $\epsilon = 0$ is stable at one loop. The angle θ_{mix} is not renormalized: its anomalous shifts are proportional to $\sum_f y_f y'_f = 0$, [86] and $B_{\mu\nu} \tilde{B}'^{\mu\nu}$ has no two-point vertex. The only renormalizable operators between the sectors are $|H|^2 |H'|^2$, $B_{\mu\nu} B'^{\mu\nu}$ and $B_{\mu\nu} \tilde{B}'^{\mu\nu}$. With $\lambda = 0$

and $\epsilon = 0$ the sectors factorize, so

$$\epsilon = \theta_{\text{mix}} = 0 \quad (76)$$

is invariant under the renormalization group to all orders and fails to be exact only through gravity. Single-graviton exchange gives zero, since $\langle J_\mu T_{\alpha\beta} \rangle = 0$, and gravity generates ϵ at six or more loops, up to $\sim 10^{-13}$ according to the estimate of Ref. [85]. States charged under both hypercharges would change this. Heavy Dirac fermions give $\epsilon \propto \sqrt{\alpha\alpha'} \sum_f y_f y'_f \ln(\Lambda/m_f)$ below their masses; for one pair with charges $(y, \pm y')$ this is the one-loop result of Ref. [87], proportional to $yy' \ln(m_2/m_1)$, which vanishes only for degenerate masses. A general spectrum can cancel without degeneracy only by tuning. Adding the swap $(P \otimes P)$ image of a bi-charged state, which keeps yy' , doubles its contribution to b_{12} unless the state is its own image.

e. The options. Combining these results, in option A, $\theta_{\text{mix},D}$ is forced by $\mathcal{M}_P = \text{swap}(P \otimes P)$, to 0 on the exact fixed locus and to $\{0, 2\pi\}$ modulo 4π up to periods, and it is not renormalized. No symmetry compatible with the chiral content forces ϵ_D ; with $\lambda = 0$ and no bi-charged states, $\epsilon_D = 0$ is radiatively stable and exact up to gravity. In reading (i) it is a radiatively stable choice; in reading (ii) it is imposed. In option B1, $\epsilon = 0$ holds exactly at M_* on the slice $\theta_Y = 0$ and is exact up to gravity below. The angle θ_{mix} is free on $\text{Fix}(\mathcal{M}_P)$ ($|\theta_{\text{mix}}| < 2\pi/3$), not renormalized, and forced to 0 only by $CP \otimes CP$: swap $(P \otimes P)$ would also force it, but together with $\mathcal{M}_P$ it requires $\alpha_Y = \alpha'_Y = 3$ at M_* , while option B1 has $\alpha_Y \leq 0.0181$ and $\alpha'_Y \geq 499$ there. In option B2, $\theta_{\text{mix},D} = 0$ at M_* and stays zero below, while ϵ_D is free at M_* and behaves below as in option A. Forcing $\epsilon = \theta_{\text{mix}} = 0$ in every per-sector frame requires $\mathcal{M}$, a C -type element or a per-sector self-duality. The first two are excluded by the chiral content, and the last needs an elliptic element on the hypercharge, hence $\alpha_Y \geq 3$, which holds only above $\mu_Y \simeq 1.3 \times 10^{41} \text{ GeV} > M_P$. If there are no global symmetries in quantum gravity [88], a conjecture proved only in AdS/CFT, the exact protections by $\mathcal{M}_P$, which contains the exchange of the sectors, and by $CP \otimes CP$ hold in a theory with gravity only if these symmetries are gauged; otherwise they are approximate. The same holds for every statement in this section that assumes an exact $\mathcal{M}_P$, in particular those of Secs. V B and V C. For P and CP , which are not internal symmetries, this reading is an extrapolation. Reference [89] argues that where spacetime reflections are symmetries in quantum gravity they must be gauged, and that with fermions this gauging is a sum over non-orientable backgrounds with pin structures. The reflection anomalies of the combined matter of the two sectors on such backgrounds [90–92], studied for the Standard Model alone in Ref. [12], are not studied here, nor are possible anomalies of the duality part $S \otimes S$ of $\mathcal{M}_P$.

f. The two readings and the millicharge. Kinetic mixing and θ_{mix} are exchanged by S , so whether ϵ counts as a portal depends on the frame. On $\text{Fix}(\mathcal{M}_P)$ ev-

ery point of the hypercharge block takes, in the frame D , the form $\hat{\tau}^D = \begin{pmatrix} x+iu & i\eta \\ i\eta & -x+iu \end{pmatrix}$, with $x = 3\theta_{Y,D}/2\pi$, $u = 12\pi/g_{Y,D}^2$ and $\epsilon_D = \eta/u$. In the original frame

$$\theta_{\text{mix}} = \frac{2\pi\eta u}{3(x^2 + u^2)}, \quad \epsilon = -\frac{\eta x}{u\sqrt{x^2 + u^2 - \eta^2}}, \quad (77)$$

so $\theta_{\text{mix}} = 0$ if and only if $\epsilon_D = 0$, and $\epsilon = 0$ if and only if $\theta_Y \theta_{\text{mix}} = 0$; a point with $\theta_{\text{mix},D} = 0$ and $\epsilon_D \neq 0$ has $\epsilon = 0$ and $\theta_{\text{mix}} \neq 0$ in the original frame. In reading (i), options A and B2 have $\epsilon_D = \pm 3\theta_{\text{mix}}/2\pi$ at $\theta_Y = 0$, $\alpha_Y \alpha'_Y = 9/(1 - \epsilon_D^2)$, and the physical ordinary coupling $g_{Y,D}/\sqrt{1 - \epsilon_D^2}$. Mirror charged states then carry the ordinary electric charge $\epsilon_\gamma Q'$, with

$$\epsilon_\gamma = -\epsilon_D \cos^2 \theta_W \quad (78)$$

for the sign convention (A6), so that $|\theta_{\text{mix}}(M_Z)| = 2.724|\epsilon_\gamma(M_Z)|$ at the common scale M_Z . In option A, where $\mathcal{M}_P$ holds at every scale, the relation to the low-energy millicharge is $|\theta_{\text{mix}}(M_Z)| = 2.918|\epsilon_\gamma(0)|$. With the same inputs, $|\epsilon_D(M_Z)| = 1.393|\epsilon_\gamma(0)|$. The factors 1.393 and 2.918 use the $\overline{\text{MS}}$ couplings $\hat{\alpha}(M_Z)$ and $\sin^2 \hat{\theta}_W(M_Z)$ of Ref. [76] and neglect the running of the off-diagonal entry $\text{Im } \hat{\tau}_{\text{mix}}^D$, which vanishes at one loop in the absence of states charged under both hypercharges (Sec. V A).

These relations turn bounds on millicharged mirror particles into bounds on the mixing parameters. The bounds below are low-energy bounds on $|\epsilon_\gamma(0)|$: in the sources, ϵ is the charge fraction of the mirror particles, i.e. the coefficient $\chi = \epsilon$ of a photon mixing $-\frac{\chi}{2}F_{\mu\nu}F'^{\mu\nu}$, the form of Eq. (A6); for the orthopositronium searches this follows from the oscillation frequency $2\pi f\epsilon$ in which they express their results [Eq. (4) of Ref. [93]]. The cosmological bound for an exact mirror parity, $|\epsilon_\gamma| < 3 \times 10^{-10}$ [94], is a model-based estimate with no confidence level attached. It assumes a mirror-to-ordinary temperature ratio $T'/T < 0.3$ at matter-radiation equality, the bound from the cosmic microwave background and large-scale structure when the dark matter consists of mirror baryons; nucleosynthesis alone gives the weaker $T'/T < 0.5\text{--}0.64$ [4, 94]. In option A, reading (i), it gives $|\epsilon_D(M_Z)| < 4.2 \times 10^{-10}$ in the frame D , where $\theta_{\text{mix},D} = 0$, and $|\theta_{\text{mix}}(M_Z)| < 8.8 \times 10^{-10}$ in the original frame, where $\epsilon = 0$ on the slice $\theta_Y = 0$. The laboratory bounds from orthopositronium, $|\epsilon_\gamma| < 3.1 \times 10^{-7}$ [95] and $|\epsilon_\gamma| < 1.55 \times 10^{-7}$ [93], both at 90% C.L., give $|\theta_{\text{mix}}(M_Z)| < 9.1 \times 10^{-7}$ and 4.5×10^{-7} ; the two searches treat collisions of orthopositronium differently, so their bounds are not directly comparable. In reading (ii), options A and B2 need $\epsilon_D = 0$, which forces $\theta_{\text{mix}} = \epsilon = 0$ in both frames and $\alpha_Y \alpha'_Y = 9$ exactly; option B1 needs only $\theta_Y \theta_{\text{mix}} = 0$, automatic on the slice $\theta_Y = 0$; and the strict form, $\epsilon = \epsilon_D = 0$, forces $\theta_{\text{mix}} = 0$ in every option. All three options are consistent in both readings, and their spectra and $n\text{--}n'$ numbers are the same, except for α'_Y in option A, which in reading (i) is $885.15/(1 - \epsilon_D^2)$.

Since ϵ and θ_{mix} conserve B and B' , neither induces $n\text{--}n'$ mixing, which needs explicit $B\text{--}L$ breaking between the sectors, such as the six-quark portal of Sec. V C; reading (i) adds only millicharged mirror states and a relative shift of order ϵ_γ of the magnetic $n\text{--}n'$ splitting.

VI. THE GEOMETRIC-LANGLANDS SIDE: HITCHIN FIBRATION, HECKE OPERATORS AND T-DUALITY

Part of our question is how the mirror map relates to T-duality and to mirror symmetry in the geometric sense, which in Ref. [10] connect the two sectors in two-dimensional models. The duality of Sec. III has such a geometric counterpart. Compactified on a Riemann surface $\mathcal{C}$, $\mathcal{N} = 4$ Yang–Mills theory with gauge group G reduces at low energies to a two-dimensional sigma model whose target is Hitchin’s moduli space $\mathfrak{M}_H(G)$ of G -Higgs bundles on $\mathcal{C}$ [19, 24, 96]. S-duality then becomes mirror symmetry between $\mathfrak{M}_H(G)$ and $\mathfrak{M}_H(G^\vee)$, realized on the generic fibers of the Hitchin fibration as a T-duality of the torus fibers together with a map of the bases [19, 25, 97, 98], in the sense of Strominger, Yau and Zaslow (SYZ) [99]. Since $G_{\text{SM}}^\vee \cong G_{\text{SM}}$, the self-duality of Sec. II should appear as an explicit auto-duality of the Hitchin fibration of G_{SM} , and $\mathcal{M}$ and $\mathcal{M}_P$ as maps of the product of the Hitchin systems of the two sectors. In this section we make both statements explicit, at the level of lattices.

We take, in each sector, the pure $\mathcal{N} = 4$ theory with gauge group G_{SM} , in which S is expected to be an exact duality: this is the Montonen–Olive conjecture [15, 17, 19, 33], extended to general gauge groups in Refs. [31, 32], and for the $\mathbb{Z}_6$ -glued group S must act on all three factors at once (App. B); Standard Model matter enters only through the labels of Wilson and ’t Hooft lines. Spacetime is $M_4 = \Sigma \times \mathcal{C}$, with $\Sigma = T^2$ (so that electric fluxes are defined) and $\mathcal{C}$ a compact Riemann surface of genus $g \geq 2$; we use $g = 1$ for the link with the two-dimensional models in Sec. VI D. All statements concern the generic fibers of the Hitchin fibration, outside the discriminant, and are made at the level of lattices and polarization types: singular fibers, spin structures and the gerbes on the non-neutral components are not treated. As in Secs. III and IV, S is a formal map. We use the complex form $S(GL_2 \times GL_3)$, which has the root datum of G_{SM} . No source treats $S(U(2) \times U(3))$ explicitly; the general results we use are those for connected reductive groups [25, 100, 101] and, for the $SU(2)$ factor, for SL_2 and PGL_2 [97].

A. The Hitchin fibration of G_{SM}

A G -Higgs bundle on $\mathcal{C}$ is a pair (E, φ) of a principal G -bundle E and a holomorphic section φ of $\text{ad}(E) \otimes K_{\mathcal{C}}$, with $K_{\mathcal{C}}$ the canonical bundle [24]. The Hitchin map

sends it to the values of a basis of invariant polynomials on φ [19, 102]. For $G_{\text{SM}} = S(GL_2 \times GL_3)$ the Higgs field is a pair $\varphi = (\varphi_2, \varphi_3)$ with $\text{tr } \varphi_2 + \text{tr } \varphi_3 = 0$, and we write φ_n^0 for the traceless parts. As a quotient, $G_{\text{SM}} = (SL_2 \times SL_3 \times \mathbb{G}_m)/\mu_6$, with $(A, B, t) \mapsto (t^3 A, t^{-2} B)$ and kernel $\{(t^{-3}, t^2, t) : t^6 = 1\}$, the complex form of Eq. (6). Its fundamental group $\pi_1(G_{\text{SM}}) = X_*/\mathbb{Z}\Phi^\vee \cong \mathbb{Z}$ is labeled by the magnetic coordinate $\tilde{Y} = \lambda_1 + \lambda_2$ of Eq. (7), and $X^*/\mathbb{Z}\Phi \cong \mathbb{Z}$ by $y = 6Y$, so that the center $Z(G_{\text{SM}}) = \mathbb{G}_m$ is connected; $\pi_1(\mathbb{G}_m) \rightarrow \pi_1(G_{\text{SM}})$ is multiplication by 6. The invariant polynomials are generated by $p_1 = \text{tr } \varphi_2$, $q_2 = \text{tr}(\varphi_2^0)^2$, $r_2 = \text{tr}(\varphi_3^0)^2$ and $r_3 = \text{tr}(\varphi_3^0)^3$, of degrees $(1, 2, 2, 3)$, and the Hitchin base is

$$\begin{aligned} \mathcal{A} &= H^0(K_C) \oplus H^0(K_C^2)^{\oplus 2} \oplus H^0(K_C^3), \\ \dim \mathcal{A} &= 12g - 11 = (g - 1) \dim G_{\text{SM}} + \dim Z(G_{\text{SM}}), \end{aligned} \quad (79)$$

so that $\dim \mathcal{A} = 13$ at $g = 2$.

a. Base map. The bases of G and $G^\vee$ are identified through an invariant bilinear form on the Cartan algebra $\mathfrak{t}$, up to a rescaling by $\mathbb{C}^\times$ [25, 98]. For a group with an abelian factor the form on the center is part of this choice. We fix it by the coupling form h of Sec. IIIB, at $\theta = 0$. The map $\Xi = 2\pi Fh : \mathfrak{t} \rightarrow \mathfrak{t}'$ is, up to sign, the unique multiple of h that is an isometry from $(\mathfrak{t}, h)$ to $(\mathfrak{t}', h')$, with h' the mirror form of Eq. (27); it acts as $4\pi/g_2^2 = g_2'/g_2$ on the $SU(2)$ Cartan, as $4\pi/g_3^2 = g_3'/g_3$ on the $SU(3)$ Cartan, and as $-12\pi/g_Y^2 = -g_Y'/g_Y$ on the hypercharge line. On the base,

$$\begin{aligned} \Xi : (p_1, q_2, r_2, r_3) &\mapsto \left(-\frac{12\pi}{g_Y^2} p_1, \left(\frac{4\pi}{g_2^2}\right)^2 q_2, \right. \\ &\quad \left. \left(\frac{4\pi}{g_3^2}\right)^2 r_2, \left(\frac{4\pi}{g_3^2}\right)^3 r_3 \right), \end{aligned} \quad (80)$$

with $g_2 g_2' = g_3 g_3' = 4\pi$ and $g_Y g_Y' = 12\pi$. The map of the mirror sector is the inverse, $\Xi(g')\Xi(g) = 1$, so that Ξ is an involution of the two-sector base; at the self-dual point $g_2 = g_3 = \sqrt{4\pi}$, $g_Y = \sqrt{12\pi}$ it reads $(-p_1, q_2, r_2, r_3)$. For the four W -equivariant isomorphisms of Sec. IIA, with signs ϵ_2, ϵ_3 , Ξ has the signs $(\epsilon_2, \epsilon_3, -\epsilon_3)$ on $(\mathfrak{su}_2, \mathfrak{su}_3, Y)$, so that only the signs of p_1 and r_3 change. Since Ξ rescales each block of the Higgs field, the spectral and cameral covers over β and $\Xi(\beta)$ are canonically isomorphic over $\mathcal{C}$.

b. Generic fiber. Over a point $\beta \in \mathcal{A}$ outside the discriminant the Higgs field defines spectral covers $\tilde{\mathcal{C}}_2$ and $\tilde{\mathcal{C}}_3$ of $\mathcal{C}$, of degrees 2 and 3 [102], and a cameral cover $\hat{\mathcal{C}}$, a W -Galois cover with $W = S_2 \times S_3$ [25, 100]; we write π_n for the projections and $\pi_{n!}$ for the Gysin (norm) maps on H^1 . The covers $\tilde{\mathcal{C}}_2$, $\tilde{\mathcal{C}}_3$ and $\hat{\mathcal{C}}$ have genera $4g - 3$, $9g - 8$ and $60g - 59$. The neutral component of the fiber is the torus $P_\beta = \Lambda_{\mathbb{R}}/\Lambda$, with lattice

$$\begin{aligned} \Lambda &= \ker (\pi_{2!} + \pi_{3!} : H^1(\tilde{\mathcal{C}}_2, \mathbb{Z}) \oplus H^1(\tilde{\mathcal{C}}_3, \mathbb{Z}) \rightarrow H^1(\mathcal{C}, \mathbb{Z})) \\ &= (H^1(\hat{\mathcal{C}}, \mathbb{Z}) \otimes X_*)^W \end{aligned} \quad (81)$$

of rank $2(12g - 11) = 2 \dim \mathcal{A}$. P_β is connected, since $\pi_{2!} + \pi_{3!}$ is onto, and

$$P_\beta = \frac{\text{Prym}(\tilde{\mathcal{C}}_2/\mathcal{C}) \times \text{Prym}(\tilde{\mathcal{C}}_3/\mathcal{C}) \times \text{Jac } \mathcal{C}}{H^1(\mathcal{C}, \mu_6)}, \quad (82)$$

through $(u'_2, u'_3, x) \mapsto (u'_2 + 3\pi_2^* x, u'_3 - 2\pi_3^* x)$, with kernel $H^1(\mathcal{C}, \mu_6) \cong (\mathbb{Z}_6)^{2g}$. This is the form of the generic fiber of a reductive group given in Ref. [25], with the μ_6 of the $\mathbb{Z}_6$ quotient. The dual lattice, $\text{Hom}(\Lambda, \mathbb{Z}) = (H^1(\tilde{\mathcal{C}}_2) \oplus H^1(\tilde{\mathcal{C}}_3))/(\pi_2^*, \pi_3^*)H^1(\mathcal{C})$, is the fiber lattice of $(GL_2 \times GL_3)/\mathbb{G}_m$, the Langlands dual in the quotient presentation of Sec. IIB. The matrix F of Eq. (2) acts on it as

$$\begin{aligned} (u_2, u_3) &\mapsto (u_2 - \pi_2^*(2c_2 - c_3), u_3 + \pi_3^*(c_2 - c_3)), \\ c_n &= \pi_{n!} u_n, \end{aligned} \quad (83)$$

and is a lattice isomorphism from $\text{Hom}(\Lambda, \mathbb{Z})$ onto the fiber lattice over $\Xi(\beta)$. Equivalently, the form $E_F(u, v) = \langle u, v \rangle - \langle \pi_{2!} u_2, \pi_{2!} v_2 \rangle_{\mathcal{C}}$, where $\langle, \rangle$ is the cup product, is unimodular on Λ ; it is the cup product tensored with the form $B^\vee = F^{-1}$ of Sec. IID, which on X_* reads $Q_5 - \tilde{Y}^2$, with Q_5 the dot product on $\mathbb{Z}^5$. For the simple factors the dual of the SL_n Prym is $\text{Prym}/J[n]$, the PGL_n fiber, as in Ref. [97] for $n = 2$. These results were computed on explicit covers at $g = 2$ (three independent ones) and at $g = 3$. For general g the lattice data rest on a topological model of the covers, a connected W -Galois cover simply branched over $4g - 4$ points (the reflection of S_2) and $12g - 12$ points (transpositions of S_3), with all such covers assumed topologically equivalent; the model agrees with the explicit covers at $g = 2$ and 3, but it is not proved for general g .

c. Polarizations. A duality of abelian varieties also concerns their polarizations. The W -invariant forms on Λ are the cup product tensored with W -invariant forms on X_* . The cup product tensored with Q_5 , $E_5(u, v) = \langle u, v \rangle$, is the restriction to Λ of the principal polarization of $\text{Jac } \tilde{\mathcal{C}}_2 \times \text{Jac } \tilde{\mathcal{C}}_3$; it decomposes as $E_5 = \mathsf{T}_2 + \mathsf{T}_3 + \frac{5}{6}\mathsf{Y}$, where T_2 and T_3 are its parts on the traceless directions of the two blocks and $\mathsf{Y}(u, v) = \langle \pi_{2!} u_2, \pi_{2!} v_2 \rangle_{\mathcal{C}}$ is the hypercharge part; then $E_F = \mathsf{T}_2 + \mathsf{T}_3 - \frac{1}{6}\mathsf{Y}$, and we define $E_+ = \mathsf{T}_2 + \mathsf{T}_3 + \frac{1}{6}\mathsf{Y}$. For every $g \geq 2$, in the topological model described below, E_5 has type $(1^{11g-11}, 5^g)$ and is auto-dual under F in no normalization; E_F is unimodular but indefinite; and E_+ is positive but not integral. The dependence on g rests here too on a topological model, that the pair $(H^1(\tilde{\mathcal{C}}_n, \mathbb{Z}), \pi_n^*)$ is unique up to isometry as a primitive embedding of $H^1(\mathcal{C}, \mathbb{Z})$ with n times its cup form into a unimodular lattice; it agrees with the explicit covers at $g = 2$ and 3 and is not proved in general. The strict-sense solutions below and Eq. (85) do not use it. The dual of a polarization E of type $(d_1, \dots, d_n)$ is $d_1 d_n E^{-1}$ [103], which F carries to $-E_F^T(d_1 d_n E^{-1})E_F$. The form $3E_+$ is integral, primitive, positive and W -invariant, of type $(1^g, 3^{10g-11}, 9^g)$, and

$$-E_F^T(9(3E_+)^{-1})E_F = 3E_+. \quad (84)$$

At the self-dual coupling F is therefore an isomorphism of polarized abelian varieties from $(P_\beta, 3E_+)$ to the dual of $(P_{\Xi(\beta)}, 3E_+)$. The positive integral W -invariant forms with this property are exactly $3kE_+$, $k \geq 1$, and $3E_+$ is the only primitive one. In the stricter sense $-E_F^T E^{-1} E_F = E$ the integral W -invariant solutions are indefinite, so that the factor $\frac{1}{3}$ in E_+ is the obstruction to an integral positive solution. Theorem A(2) of Ref. [25] asserts a polarized isomorphism for simple G without fixing the normalization of the dual polarization; read with the dual of Ref. [103], Eq. (84) is the corresponding statement for G_{SM} .

The couplings enter through the class $R(t) = T_2/t_2 + T_3/t_3 + Y/(6t_Y)$, with $g_k^2 = 4\pi t_k$ ($k = 2, 3$) and $g_Y^2 = 12\pi t_Y$, so that $E_+ = R(1)$ is the class at the self-dual point. It satisfies

$$-E_F^T R(t)^{-1} E_F = R(1/t) : \quad (85)$$

F carries the dual of the fiber at the couplings t to the fiber at the couplings $1/t$, which are those of Eq. (27).

d. Components. The fiber has infinitely many components. The components of the generic fiber, and of $\mathfrak{M}_H(G_{\text{SM}})$, are labeled by $\pi_0 = \pi_1(G_{\text{SM}}) \cong \mathbb{Z}$, with label the degree $d = \deg E_2$ of the $U(2)$ block, which is the magnetic coordinate $\tilde{Y}$; each component is a torsor under the connected group P_β , and there is no finite group of components, since $Z(G_{\text{SM}})$ is connected and $\pi_1(G_{\text{SM}})$ is torsion free. The identification $\pi_0(\mathfrak{M}_H(G)) = \pi_1(G)$ for a reductive group G is proved in Refs. [25, 101]; here it is recomputed in the spectral description. On the dual side the components of the fiber of $(GL_2 \times GL_3)/\mathbb{G}_m$ form $\mathbb{Z}^2/(2, 3)\mathbb{Z} \cong \mathbb{Z}$, with label $e = 3d_2 - 2d_3 = y$, the characters of $Z(G_{\text{SM}})$, and F sends $e \mapsto d = -e$, as in Eq. (9). The finite part of the structure is the kernel $H^1(\mathcal{C}, \mu_6) \cong (\mathbb{Z}_6)^{2g}$ of Eq. (82).

B. Flux sectors

A G -bundle on M_4 has a characteristic class $\xi \in H^2(M_4, \pi_1(G))$, and the partition functions Z_ξ transform in a representation of the duality group [17, 19]. In the Hamiltonian picture on $S_x^1 \times \mathcal{C}$ the states carry a magnetic flux m and an electric flux e , and S acts as $(e, m) \mapsto (-m, e)$ [19]. These statements assume a finite center. For G_{SM} , $\pi_1 = \mathbb{Z}$ and $Z(G_{\text{SM}}) = U(1)$, and we use instead the pairing between $\pi_1(G^\vee)$ and the characters of $Z(G)$ of Ref. [25]; this extension is our interpretation. Per sector the flux labels are $(\xi_m, \xi_e) \in \mathbb{Z}^2 \otimes H^2(M_4, \mathbb{Z})$, of rank $2(4g + 2)$, with $H^2(M_4) = H^2(\Sigma) \oplus H^1(\Sigma) \otimes H^1(\mathcal{C}) \oplus H^2(\mathcal{C})$, the mixed components included; on $S_x^1 \times \mathcal{C}$ they are $(m, e) \in \mathbb{Z}^2 \otimes H^2(S_x^1 \times \mathcal{C})$, the charges of the 1-form symmetries $U(1)_m$ and $U(1)_e$ of Sec. IV B, and the degree is m_0 , the $H^2(\mathcal{C})$ component of m . On the coefficients $(m, e) = (\tilde{Y}, y)$ the duality group

acts by

$$\begin{aligned} \rho(S_\Phi) &= \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \rho(\delta(T)) &= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \\ \rho(P) &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, & \rho(C) &= -1, \end{aligned} \quad (86)$$

with the upper sign for Φ in class B, where $\rho(S_\Phi)$ is the S of Eq. (A2), $(e, m) \mapsto (-m, e)$, and the lower sign for class A, which contains F . More generally $\rho(\delta(\gamma)) = \gamma^{-T}$ for all $\gamma \in SL(2, \mathbb{Z})$, with δ the diagonal embedding of Eq. (35); W and $\Gamma(3) \times \Gamma(2)$ act trivially, ρ is a homomorphism with image $GL(2, \mathbb{Z})$, and $S_\Phi^2 = C$, $PSP = S^{-1}$ and $PTP = T^{-1}$ hold as in Eq. (22). On the full lattices an orientation-preserving element acts as $\rho \otimes 1$ and P as $\rho(P) \otimes (-r^*)$, where r is an orientation-reversing involution of M_4 : the magnetic flux is a cohomology class and the electric flux a pseudo-class. We consider two choices, r_x ($x \rightarrow -x$ on Σ) and r_C (an antiholomorphic involution of $\mathcal{C}$). No element mixes the cohomology components.

On the two sectors, with the labels ordered as (m_1, m_2, e_1, e_2) , each of the $24 \times 24 \times 2$ elements $X = \text{swap}(P \otimes P)^{0,1}(S_\Phi \otimes S_\Phi)$ of Sec. III E maps $m_1 \rightarrow \pm e_2$, $e_1 \rightarrow \pm m_2$, $m_2 \rightarrow \pm e_1$ and $e_2 \rightarrow \pm m_1$, with signs that depend only on the classes of Φ and Φ' and on whether P is present. For Φ, Φ' in class A, for example,

$$\rho(\mathcal{M}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \rho(\mathcal{M}_P) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (87)$$

For all elements $\mathcal{M}^2 = \tilde{\mathcal{M}}_P^2 = -1$ and $\mathcal{M}_P^2 = \tilde{\mathcal{M}}^2 = +1$ on the flux labels, and $\rho J \rho^T = +J$ for $\mathcal{M}, \tilde{\mathcal{M}}$ and $-J$ for $\mathcal{M}_P, \tilde{\mathcal{M}}_P$, where J is the Dirac pairing, as in Eq. (23) and Table II.

a. Degree and electric grading. The duality maps the components of one Hitchin system to the electric grading of the other. Under $\mathcal{M}$ and $\mathcal{M}_P$ the degree $d = Y \in \pi_0(\mathfrak{M}_H(G_{\text{SM}}))$ goes to the character y' of $Z(G'_{\text{SM}})$,

$$\begin{aligned} d \mapsto y' &= -\frac{5}{c_\Phi} d, & \Phi K \lambda_c &= c_\Phi \lambda_c, \\ c_\Phi &= -5 \ (\Phi \in \text{A}), & c_\Phi &= +5 \ (\Phi \in \text{B}), \end{aligned} \quad (88)$$

where K is the trace-form Gram matrix of Eq. (11) and λ_c the central cocharacter. The factor $c_\Phi = -5$ of class A is the -5 of Eq. (11). It is canceled by the index $y(K\lambda) = 5\tilde{Y}(\lambda)$ of the trace-form identification, so the integral map is $d \mapsto y' = +d$ for class A and $-d$ for class B, for $\mathcal{M}$ and for $\mathcal{M}_P$ with $r = r_C$; for $\mathcal{M}_P$ with $r = r_x$ the sign is reversed. No integral map with a factor -5 exists, since an isomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ is ± 1 : the -5 lives in the identification of the Lie algebras by the trace form. In the other direction $\tilde{Y}' = \pm(c_\Phi/5)y$, which is Eq. (55). On the Hamiltonian lattice at $g = 2$ the degree m_0 goes only to e'_0 , the $H^2(\mathcal{C})$ component of the mirror electric flux, and the mixed components go to the same components.

b. Consistency with the line maps. Line operators shift the flux sectors [19]. A line of charge q along a 1-cycle c of $S_x^1 \times \mathcal{C}$ shifts the labels by $v(q) \otimes \text{PD}(c)$, with $v(q) = (\tilde{Y}, y)$: a Wilson line shifts e and an 't Hooft line shifts m . The shifts are equivariant, $v(qX^{-1}) = \rho(X)v(q)$, for all 1152 elements, on all unit charges and on all Standard Model weights of both sectors. The representation ρ reproduces the flux classes of Eq. (55) and is the inverse of the transport of the 1-form symmetry charges of Eq. (57), with all its sign patterns; the anomaly coefficient goes to $+K$ under $\mathcal{M}, \tilde{\mathcal{M}}$ and to $-K$ under $\mathcal{M}_P, \tilde{\mathcal{M}}_P$. A shift $\Delta\theta_{\text{mix}} = 4\pi$ acts as $e_1 \rightarrow e_1 - 6m_2$, $e_2 \rightarrow e_2 - 6m_1$. In the frames D and D' of Table II every element acts classically on the flux labels, as ρ of its classical form.

C. Hecke modifications and Wilson lines

On the Hitchin moduli space an 't Hooft operator with magnetic label $m \in X_*$ acts as a Hecke modification of type m at a point $p \in \mathcal{C}$ [19, 33, 104]. For GL_N , a coweight $\mu = (\mu_1, \dots, \mu_N)$ acts on a local splitting $E = \bigoplus_i L_i$ as $L_i \rightarrow L_i(\mu_i p)$ and shifts the degree by $\sum_i \mu_i$ [19]. For $S(GL_2 \times GL_3)$ we apply this block by block to $E_2 = L_1 \oplus L_2$ and $E_3 = L_3 \oplus L_4 \oplus L_5$, with $\deg E_2 + \deg E_3 = 0$; this form is our interpretation. In the $\mathbb{Z}^5$ form $\mu = (\mu_1, \mu_2 | \mu_3, \mu_4, \mu_5)$, $\sum_i \mu_i = 0$, the modification shifts the component label $d = \deg E_2$ by $\Delta d = \mu_1 + \mu_2 = \tilde{Y}(\mu)$.

Table VII lists, for each Standard Model representation R with highest weight λ , the Hecke modification of G'_{SM} labeled by the coweight $F\lambda$. The electron's Wilson line corresponds to the invertible, central modification $E_2 \rightarrow E_2(3p)$, $E_3 \rightarrow E_3(-2p)$, which raises the degree by 6. In general,

$$\Delta d = \tilde{Y}(F\lambda) = -y(\lambda) \quad \text{for all } \lambda \in X^*, \quad (89)$$

and roots go to coroots, with $\Delta d = 0$. The four generators of $\Lambda_e = X^*$ give $\Delta d = -3, -3, 2, 2$, whose greatest common divisor is 1, so every component is reached. This is where the -5 of Eq. (10) goes: with ι the trace-form identification of $\mathfrak{t}$ with $\mathfrak{t}^*$, $F\iota = 1$ on the coroots and $F\iota = -5$ on the central cocharacter, so the hypercharge component of $\iota^{-1}\lambda$ is $y/5$ and that of $F\lambda$ is $-y$. The factor 5 is absorbed by the trace form, and the integral degree map is Eq. (89): a Wilson line of hypercharge Y corresponds to a Hecke modification that changes the degree by $-6Y$.

The table is consistent with the geometric Satake correspondence [98, 105]. All Standard Model representations are minuscule, and F maps the weights of R one to one onto the W' -orbit of μ , which is the set of fixed points of the maximal torus on the Grassmannian $\text{Gr}_\mu = G'/P_\mu$. Moreover $\dim \text{Gr}_\mu = \langle 2\rho', \mu \rangle$, with ρ' the half sum of the positive roots of G'_{SM} ; $\chi(\text{Gr}_\mu) = \dim R$; and the Betti numbers of Gr_μ are the principal grading of R .

a. Two sectors. On $G_{\text{SM}} \times G'_{\text{SM}}$ the dictionary is fixed by $\mathcal{M}$ and $\mathcal{M}_P$. Under each of the 1152 elements X an ordinary Wilson line of weight ν goes to a pure mirror 't Hooft line, i.e. to a Hecke modification of G'_{SM} , of coweight

$$m_2 = s \Phi \nu, \quad (90)$$

with $s = +1$ for the elements without P ($\mathcal{M}, \tilde{\mathcal{M}}$) and $s = -1$ for those with P ($\mathcal{M}_P, \tilde{\mathcal{M}}_P$), for Φ in either class; mirror Wilson lines go to ordinary 't Hooft lines in the same way, $m_1 = s \Phi' \nu'$. The type of the modification is $\mu(R) = F\lambda_R$ of Table VII for Φ in class A without P and in class B with P , and $\mu(\tilde{R})$ otherwise. For $\Phi = F$ and $\mathcal{M}_P$, for example, Q_L goes to $\mu = (1, 0 | 0, 0, -1)$, i.e. $E_2 \subset E'_2 \subset E_2(p)$ and $E_3(-p) \subset E'_3 \subset E_3$, with $\Delta d' = +1$. The degree shift in $\pi_0(\mathfrak{M}_H(G'_{\text{SM}}))$ is $\Delta d' = \tilde{Y}(m_2)$, equal to $-y$ for class A and $+y$ for class B without P , and to the opposite with P ; these are the flux classes of Eq. (55), so the Hecke dictionary and the line maps of Sec. IV B agree.

The map has the form of the Hecke eigen-property, at the level of lattices on generic fibers. A Wilson operator W_R at p acts on a generic fiber as $\bigoplus_{\nu \in \text{wt}(R)} \text{Tens}_\nu$, tensoring with line bundles, and shifts the electric label by y [19]. The element X conjugates it into $\bigoplus_\nu \text{Trans}_{s\Phi\nu}$, translations by the T' -fixed points of Gr_μ , each once, so that there are $\dim R = \chi(\text{Gr}_\mu)$ terms, all with the same component shift $\Delta d' = \tilde{Y}(\mu)$. This is the abelianized form of the statement that a Hecke eigensheaf $\mathcal{F}$ satisfies $T_\mu \mathcal{F} = \mathcal{F} \boxtimes V_R$, in the exchange of translation and tensorization operators of Ref. [25]; neither the Fourier–Mukai transform nor the universal bundle is constructed here.

D. Product T-duality

a. One sector. Over a generic β the two-dimensional zero modes of one sector form the lattice $V = \Lambda \oplus \Lambda^\vee$, with windings in $\Lambda = (H^1(\hat{\mathcal{C}}) \otimes X_*)^W$ [Eq. (81)] and momenta in $\Lambda^\vee = (H^1(\hat{\mathcal{C}}) \otimes X^*)^W$, and the pairing $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We let an element of the charge lattice with W -equivariant blocks act on V as $1 \otimes \gamma$, which is our reading of the abelianized four-dimensional charges on the cameral cover. T-duality of the whole fiber torus, $\text{FM}(w, n) = (n, w)$, is the fiberwise Fourier–Mukai transform at the level of zero modes [19, 106]. Then S_F acts on V as

$$S_F = \text{FM} \circ F : (w, n) \mapsto (Fn, F^{-T}w), \quad (91)$$

that is, $S_F = \text{diag}(F, F^{-T}) \text{FM}$, where $F = E_F^{-1}$ is the isomorphism of Eq. (83) onto the lattice over $\Xi(\beta)$ and $F^{-T} = -E_F$. It preserves η and squares to -1 . The other W -equivariant choice of class A, sF with s the Weyl reflection of $SU(2)$, gives $\text{FM} \circ F \circ (\sigma \oplus \sigma^\vee)$, where σ is the deck involution of $\tilde{\mathcal{C}}_2 \rightarrow \mathcal{C}$; it acts as -1 on $\text{Prym}(\tilde{\mathcal{C}}_2/\mathcal{C})$, i.e. as $\varphi_2 \rightarrow -\varphi_2$.

TABLE VII. Hecke modifications of G'_{SM} that correspond under F to the Wilson lines of one Standard Model generation and of the Higgs doublet. For each highest weight $\lambda = (a_1, a_2, n, y)$ (Table I) the coweight $F\lambda$ has the $\mathbb{Z}^5$ form μ ; the modification at $p \in \mathcal{C}$ is given on the two blocks, with the Grassmannian of choices where there is one, and Δd is the shift of the degree $d = \deg E_2$.

Field	λ	$F\lambda$	μ	E_2	E_3	Δd
Q_L	(1, 0, 1, 1)	(1, 0, 1, -1)	(0, -1 1, 0, 0)	$E_2(-p) \subset E'_2 \subset E_2, \mathbb{P}^1$	$E_3 \subset E'_3 \subset E_3(p), \mathbb{P}^2$	-1
u_R	(1, 0, 0, 4)	(1, 0, 0, -4)	(-2, -2 2, 1, 1)	$E_2 \rightarrow E_2(-2p)$	$E_3(p) \subset E'_3 \subset E_3(2p), \mathbb{P}^2$	-4
d_R	(1, 0, 0, -2)	(1, 0, 0, 2)	(1, 1 0, -1, -1)	$E_2 \rightarrow E_2(p)$	$E_3(-p) \subset E'_3 \subset E_3, \mathbb{P}^2$	2
L_L	(0, 0, 1, -3)	(0, 0, 1, 3)	(2, 1 -1, -1, -1)	$E_2(p) \subset E'_2 \subset E_2(2p), \mathbb{P}^1$	$E_3 \rightarrow E_3(-p)$	3
e_R	(0, 0, 0, -6)	(0, 0, 0, 6)	(3, 3 -2, -2, -2)	$E_2 \rightarrow E_2(3p)$	$E_3 \rightarrow E_3(-2p)$	6
H	(0, 0, 1, 3)	(0, 0, 1, -3)	(-1, -2 1, 1, 1)	$E_2(-2p) \subset E'_2 \subset E_2(-p), \mathbb{P}^1$	$E_3 \rightarrow E_3(p)$	-3

b. Two sectors. On $V \oplus V'$ the mirror elements become T-dualities of the product. For Φ, Φ' in class A,

$$\mathcal{M} = \text{swap} \circ (\text{FM} \circ F \oplus \text{FM} \circ F), \quad \mathcal{M}^2 = -1, \quad (92)$$

and $-\mathcal{M}$ for class B; $\mathcal{M}$ lies in $O(V \oplus V', \eta)$. With P realized by r_x , P acts on V as the worldsheet parity $\Omega = \text{diag}(-1_\Lambda, 1_{\Lambda^\vee})$, and $\mathcal{M}_P = (\Omega \otimes \Omega)\mathcal{M}$, with $\mathcal{M}_P^2 = 1$ and $\eta \rightarrow -\eta$; moreover $\tilde{\mathcal{M}}^2 = 1$ and $\tilde{\mathcal{M}}_P^2 = -1$. Every element maps windings onto momenta, and its labels $(\tilde{Y}, y) \otimes H^1(\mathcal{C})$ transform by $\rho \otimes 1$ of Eq. (86). On the base it acts as $(\beta, \beta') \mapsto (\Xi(\beta'), \Xi(\beta))$, and the degree goes as in Eq. (88). The reflection r_x is the one that becomes the worldsheet parity of Sec. III F, and it needs no real structure on $\mathcal{C}$.

c. Sigma model. Reduced on $\mathcal{C}$, the product theory is a two-dimensional sigma model on Σ with the hyperkähler target $\mathfrak{M}_H(G_{\text{SM}}) \times \mathfrak{M}_H(G'_{\text{SM}})$ [19, 98]. We describe it by its semi-flat zero modes at $\theta = 0$. The fiber zero modes $V \oplus V'$ form an even unimodular lattice for η . The fiber metric is $G_t = R(t)J$, with $R(t)$ the class of Eq. (85) and J any complex structure compatible with $\mathbf{T}_2, \mathbf{T}_3$ and $\mathbf{Y}$, and the generalized metric is $\mathcal{H}_t = \text{diag}(G_t, G_t^{-1})$. The map F is holomorphic, $FJ^{-T} = JF$, and $\text{FM} \circ F$ maps the model at t over β to the model at $1/t$ over $\Xi(\beta)$, $S_F^T \mathcal{H}_{1/t} S_F = \mathcal{H}_t$. For $X = \mathcal{M}, \mathcal{M}_P$ (with r_x), and $\mathcal{M}$ built from sF ,

$$X^T(\mathcal{H}_t \oplus \mathcal{H}_{t'})X = \mathcal{H}_t \oplus \mathcal{H}_{t'} \iff t' = 1/t, \quad (93)$$

that is, $g_2 g'_2 = g_3 g'_3 = 4\pi$ and $g_Y g'_Y = 12\pi$, with $\beta' = \Xi(\beta)$. On this locus $\mathcal{M}$ is a T-duality automorphism of the sigma model, the analogue of the decoupled condition $R_1 R_2 = 1$ of Sec. III F. The target is self-mirror: its SYZ mirror $\mathfrak{M}_H(G_{\text{SM}}^\vee) \times \mathfrak{M}_H(G'_{\text{SM}}^\vee)$ is identified with it by F and Ξ . Only the complex structure I enters these statements; the exchange of A- and B-models in the other complex structures is taken from Refs. [19, 98, 104].

d. The non-abelian blocks. The zero-mode lattice splits along the factors of Eq. (82). V is an overlattice of three η -orthogonal blocks, in each of which the momenta are those orthogonal to the windings of the others: an abelian block V_{ab} , with windings $\lambda_c \otimes H^1(\mathcal{C})$ and momenta $\ell_Y \otimes H^1(\mathcal{C})$, of ranks $(2g, 2g)$ and discriminant group $(\mathbb{Z}_6)^{4g}$; a block $V_{SU(2)}$, with windings the SL_2

Prym lattice $\Lambda_{SL_2} = \ker \pi_{2!}$, of ranks $(6g - 6, 6g - 6)$ and discriminant group $(\mathbb{Z}_2)^{4g}$; and a block $V_{SU(3)}$, with windings $\Lambda_{SL_3} = \ker \pi_{3!}$, of ranks $(16g - 16, 16g - 16)$ and discriminant group $(\mathbb{Z}_3)^{4g}$. The glue,

$$V/(V_{ab} \oplus V_{SU(2)} \oplus V_{SU(3)}) \cong (\mathbb{Z}_6)^{4g}, \quad (94)$$

projects onto each discriminant group, so it identifies the discriminant group of V_{ab} with that of $V_{SU(2)} \oplus V_{SU(3)}$; it is the lattice form of $H^1(\mathcal{C}, \mu_6)$, i.e. of the $\mathbb{Z}_6$ quotient. Every element maps each block of one sector onto the same block of the other, windings to momenta, so that $\mathcal{M} = \text{swap} \circ \bigoplus_x (\text{FM} \circ F|_x)$ on the glued lattice, $x \in \{ab, SU(2), SU(3)\}$. Inside $V_{SU(n)}$ the momenta form a sublattice of index n^{2g} in $\text{Hom}(\Lambda_{SL_n}, \mathbb{Z})$, the PGL_n Prym lattice that is the dual of the SL_n Prym [97]: on a non-abelian block alone, $\text{FM} \circ F$ maps SL_n Prym windings to SL_n -type momenta, not onto the full PGL_n dual. No block is unimodular by itself, and only the μ_6 -glued V is self-dual and invariant under $\mathcal{M}$. The fixed locus splits by blocks: $g_2 g'_2 = g_3 g'_3 = 4\pi$ on the non-abelian blocks, and $g_Y g'_Y = 12\pi$ with $\epsilon = 0$ on the abelian one. We read the non-abelian part as the SYZ T-duality of the Prym parts of the two Hitchin fibrations, glued to the abelian compact-boson T-duality by $H^1(\mathcal{C}, \mu_6)$; this reading is structural and not computed. It holds only over generic β , it has no free-boson description, and its status as an exact two-dimensional symmetry rests on S being an exact duality.

e. The abelian block and SMM2b. The abelian block is the bridge to the two-dimensional models of Sec. III F. We take the two hypercharge factors with integer charges $y = 6Y$, coupling $\tau_y = 6\hat{\tau}_Y$ ($\theta_y = 18\theta_Y$), and the mixed coupling $\hat{\tau}_{\text{mix}}$ of Eq. (39), and reduce the four-dimensional Lagrangian on $S_x^1 \times \mathcal{C}$ with $\mathcal{C} = T^2$ ($g = 1$), with the charge labels of Sec. IV. At $g = 1$ only this abelian block is used; the Hitchin base of Eq. (79) and its dimension $12g - 11$ refer to $g \geq 2$. The flux sectors along the cycles a, b of $\mathcal{C}$ are two compact bosons per sector, $X_c = \oint_c A_\mu dx^\mu$, whose zero-mode energy is that of the model (44), $\frac{\pi}{L_x} q^T \mathcal{H}(E) q$, with $E = G + B$, $G = \text{Im } \tau \otimes \text{diag}(L_b/L_a, L_a/L_b)$ and $B_{ba} = -B_{ab} = \text{Re } \tau$, for cycle lengths L_a, L_b and circle length L_x . Per cycle the dictionary between four-dimensional charges (n_m, n_e)

and windings and momenta is

$$(n_m, n_e)_a = (w_b, n_a), \quad (n_m, n_e)_b = (-w_a, n_b), \quad (95)$$

and the two-dimensional pairing is $\eta = \omega_C \otimes J$, with ω_C the intersection form on $H_1(C)$ and J the Dirac pairing. In the lattice labels the background is $E(\hat{\tau})$, with an overall factor $\frac{1}{6}$ from the μ_6 glue. The reduction is equivariant for all elements, with P realized by r_x or by r_C . Since ω_C and J are both antisymmetric, η is symmetric, and $\mathcal{M}$, which preserves the Dirac pairing, preserves η ; $\mathcal{M}_P$ reverses η with r_x and preserves it with r_C . Unlike the identification of Sec. III F, which was a choice, Eq. (95) follows from the reduction.

For a square $\mathcal{C}$ at $\theta = 0$, each element composed with the rotation $a \rightarrow -b$, $b \rightarrow a$ of $\mathcal{C}$ preserves the pair of bosons (X_a, X'_a) and is an element of Sec. III F. For the classes BB,

$$\begin{aligned} \mathcal{M} &\rightarrow M_{2D}, & \mathcal{M}_P(r_x) &\rightarrow M_{P,2D}, \\ \mathcal{M}_P(r_C) &\rightarrow M_{2D}(C \otimes C), \end{aligned} \quad (96)$$

and for the classes AA the same elements times $C \otimes C$, which is central and acts trivially on the backgrounds; for the mixed classes BA, $\mathcal{M} \rightarrow \tilde{M}_{2D}$ and $\mathcal{M}_P(r_x) \rightarrow \tilde{M}_{P,2D}$, and for AB the same times $C \otimes C$. The opposite rotation multiplies every entry by $C \otimes C$. With $(G_{11}, G_{22}, G_{12}) = \text{Im } \hat{\tau}$ and $b = 0$, the fixed loci of Sec. III F, $G_{12} = 0$ and $G_{11}G_{22} = 1$ for M_{2D} and $M_{P,2D}$, and $\det G = 1$ for the lifts, are those of Sec. III D: $\epsilon = 0$ and $g_Y g'_Y = 12\pi$ for $\mathcal{M}$ and $\mathcal{M}_P$, and $\det \text{Im } \hat{\tau} = 1$ with ϵ free for $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$. For $g = 1, 2$ at $\theta = 0$, and any compatible complex structure on $\mathcal{C}$, $\mathcal{M}$ is a symmetry of the Jac $\mathcal{C}$ model if and only if $\epsilon = 0$ and $\text{Im } \hat{\tau}_{11} \text{Im } \hat{\tau}_{22} = 1$; at $\theta_Y \neq 0$ this condition is not sufficient. The bosons are free, since $\text{Jac } \mathcal{C}$ is flat, as the T-duality of the compact multiplet of SMM2b requires (no potential, Sec. III F). The abelian block of the product T-duality is thus, at the level of lattices, the compact-boson T-duality of SMM2b with one boson per sector [10]. The match is bosonic: the $\mathcal{N} = (1, 1)$ fermions of Sec. III F are not compared with the fermions of the $\mathcal{N} = 4$ theory.

The extension of this construction to the gauge group of the four-dimensional mirror models is left to the companion paper [30].

In summary, the self-duality of G_{SM} is an auto-duality of its Hitchin fibration: F identifies the generic fiber with the dual of the fiber over $\Xi(\beta)$, the mirror elements act on the product of the two Hitchin systems, at the level of zero-mode lattices on generic fibers, as the sector exchange composed with fiberwise T-duality, and the abelian part of this T-duality is the compact-boson T-duality of the two-dimensional mirror models. For mirror matter, this places the T-duality that relates the sectors in the two-dimensional models of Ref. [10] inside the four-dimensional construction; the geometry involved is the hyperkähler Hitchin moduli space of each sector, not a Calabi–Yau threefold.

VII. CONCLUSIONS AND OUTLOOK

We have asked whether the self-duality of the Standard Model gauge group can be used to construct the map between the ordinary and mirror sectors of mirror matter. The first part of the answer is group theoretic: $G_{\text{SM}} = S(U(2) \times U(3))$ is Langlands self-dual, through explicit isomorphisms of root data, of groups and of charge lattices, and among the global forms of the Standard Model group with the observed matter it is the only one with this property. The property is not generic: among the groups $S(U(n_1) \times \cdots \times U(n_k))$ it requires a congruence condition on the block sizes, which for two blocks holds only when one block has size 1 or the sizes differ by 1.

The second part is that the self-duality does not by itself produce a second sector; the product construction does, and it does not need a self-dual group. What the self-duality adds comes down to three things: the electric-frame group of the mirror sector is again G_{SM} , the two sectors share one lattice of θ periods, and the dictionary carries a charge-conjugation twist, the price of the identification. That a sector exchange and an S-type self-map of each sector coexist, and that a single sector has a point that is self-dual under an S-type map on all three factors, restate the existence of the root-datum isomorphism; once T shifts are allowed, the non-self-dual quotient by $\mathbb{Z}_3$ also has an order-4 self-duality on all three factors. That the mirror matter is magnetic is forced by the matter, and without self-duality by the gauge group as well. With S treated as a formal map, the element $\mathcal{M}_P = \text{swap} \circ P \circ (S \otimes S)$ fixes the couplings of $T[G_{\text{SM}}, \tau] \otimes T[G'_{\text{SM}}, \tau']$ on the locus $\tau' = 1/\bar{\tau}$, i.e. $g_3 g'_3 = g_2 g'_2 = 4\pi$ and $g_Y g'_Y = 12\pi$ at $\theta = 0$. Its physical status depends on S. If per-sector S-duality is an exact equivalence, $\mathcal{M}_P$ is the standard mirror parity written in an electric frame for the ordinary sector and a magnetic frame for the mirror sector, and the theory has the observables of conventional mirror matter; if it is not, as App. B indicates for the chiral Standard Model, $\mathcal{M}_P$ is not a symmetry. The statements that do not depend on this choice of frame are listed in Sec. III H. With one chiral generation in each sector, $\mathcal{M}_P$ is compatible with the matter and exchanges left-handed ordinary with right-handed mirror fermions; this helicity flip comes from parity alone, and the C-twist that the $\mathbb{Z}_6$ glue forces into the dictionary does not produce it. On line operators the duality maps ordinary Wilson lines to mirror 't Hooft lines, so that it can be a symmetry with matter only if the mirror matter is magnetic in the electric frame of the mirror sector: the electron's Wilson line goes to the minimal pure-hypercharge monopole line of the mirror sector. This is the precise content of the statement “ordinary electric $\leftrightarrow$ mirror magnetic”.

For phenomenology the main consequence is that the two electric-frame couplings are inversely related. On the fixed locus of $\mathcal{M}_P$, $\alpha_3 \alpha'_3 = \alpha_2 \alpha'_2 = 1$ and $\alpha_Y \alpha'_Y = 9$, and the Standard Model reaches the self-dual values at

no scale between about 1 GeV and the Planck mass. In option A, $\mathcal{M}_P$ is exact and the mirror sector is an exact parity image in its S-dual frame, with degenerate n and n' ; this requires per-sector S-duality of the chiral Standard Model, which is not established. In option B1 the relation is imposed only at an ultraviolet scale M_* with matter electric in both sectors, where $\mathcal{M}_P$ is not a symmetry, so that it is a boundary condition without a symmetry behind it; mirror color then confines near M_* , the mirror neutron is heavy, and n - n' oscillations are negligible. In option B2 the theory above M_* is S-dual, which needs the same unestablished S-duality with chiral matter as option A, the mirror matter is electric in the S-dual frame, and the duality is broken below M_* ; a mirror neutron within a few μeV of the neutron then requires an extreme tuning of the couplings of the two sectors. For the Higgs-vev asymmetry $(v' - v)/v \sim 10^{-15}$ – 10^{-14} of Refs. [6, 9], the mismatch of α_3^{-1} between the sectors must be of the same order or smaller so as not to dominate the splitting, as in any mirror model whose exchange symmetry is broken softly.

For n - n' oscillations, described as in Ref. [9] by a mass splitting Δ and a mixing angle $\theta_{nn'}$, an exact $\mathcal{M}_P$ forces equal flavor-basis masses, $\Delta \cos 2\theta_{nn'} = 0$, so that $\theta_{nn'} = \pi/4$ and $\Delta = 2|\delta_m|$ for any mixing mass δ_m , as for exact mirror parity [23]: this is the degenerate regime assumed by the storage experiments, and the nondegenerate region of Ref. [9] requires $\mathcal{M}_P$ to be broken. If it is broken only by the Higgs-vev asymmetry, the flavor-basis splitting is $\Delta \cos 2\theta_{nn'} = \kappa m_n (v' - v)/v$, with a hadronic factor κ that depends on how the breaking reaches the QCD scale. The mixing angle itself is a phenomenological input: without portal operators, δ_m has no source in any topological sector of any global form on closed spin backgrounds, with or without $\mathcal{M}_P$; backgrounds with boundaries, non-orientable backgrounds, wormholes and black holes are not covered. For the portals between the sectors, which we classify completely up to dimension 6, $\mathcal{M}_P$ has the selection rules of exact mirror parity in the S-dual frame: it forbids exactly the mixed θ -angle of that frame and the dimension-6 neutrino portal $(LH)(L'D_\mu H')$, the latter for every lift of $\mathcal{M}_P$ that squares to +1 on the lepton doublets and already forbidden by the accidental $B - L$ symmetries, and it constrains all others. The first vanishes to all orders while $\mathcal{M}_P$ is exact; the neutrino and six-quark portals change $B - L$ and, set to zero together, are regenerated only by gravity, while the Higgs portal that $\mathcal{M}_P$ allows is regenerated at one loop by kinetic mixing. The other element compatible with chiral matter, $\mathcal{M}$, forbids no portal. With Standard Model matter only $\bar{\theta}$ and an electroweak combination I_{EW} of the θ -angles and Yukawa phases are physical in each sector. If S is exact and $\mathcal{M}_P$ is extended to the Yukawa couplings by parity, its fixed set has $\bar{\theta}' = -\bar{\theta}$ in a common orientation of the two sectors (copy-type exchange symmetries give $\bar{\theta}' = +\bar{\theta}$ instead [48, 49]), so the mirror sector has a strong-CP problem of the same size; in the S-dual frame this is the relation that the standard mirror parity im-

poses, and what the construction adds is its reading in the original frame. If $\mathcal{M}_P$ acts on the gauge couplings only, it constrains $\bar{\theta}$ and $\bar{\theta}'$ only through a choice of slice. For kinetic mixing, no symmetry compatible with the chiral content forces $\epsilon = 0$, while $\mathcal{M}_P$ forces the mixed angle of the S-dual frame to vanish in that frame; without Higgs-portal couplings and states charged under both hypercharges, $\epsilon = \theta_{\text{mix}} = 0$ is stable to all orders and broken only through gravity. If the mixing is nonzero, in option A the cosmological bound on millicharged mirror particles, which assumes a mirror sector colder than ours, $T'/T < 0.3$, gives $|\epsilon_D(M_Z)| < 4.2 \times 10^{-10}$ in the S-dual frame, or $|\theta_{\text{mix}}(M_Z)| < 8.8 \times 10^{-10}$ in the original one.

On the geometric side the construction is related to T-duality and to mirror symmetry in the sense of Strominger, Yau and Zaslow [99], but the target is the hyperkähler Hitchin moduli space of each sector, not the Calabi–Yau threefold of the compactification of Ref. [10]; the Calabi–Yau part of our original question is therefore not answered here. On the Hitchin moduli spaces of the two sectors the self-duality is an auto-duality of the fibration of G_{SM} , and $\mathcal{M}$ acts as the exchange of the sectors composed with fiberwise T-duality, the lattice form of a fiberwise Fourier–Mukai transform; on the coupling locus above, it is a T-duality of the two-dimensional sigma model with the self-mirror target $\mathfrak{M}_H(G_{\text{SM}}) \times \mathfrak{M}_H(G'_{\text{SM}})$. Its abelian part is the compact-boson T-duality of the two-dimensional mirror model SMM2b of Ref. [10], and the two-dimensional mirror maps built from T-duality form the same group as $\mathcal{M}$ and $\mathcal{M}_P$, with the roles of orientation-preserving and orientation-reversing elements exchanged. These statements hold for generic fibers and at the level of lattices; the Fourier–Mukai transform itself and the singular fibers are not treated.

Several questions remain open. The first is the status of S-duality for G_{SM} with chiral matter. S is expected to hold for $\mathcal{N} = 4$ super–Yang–Mills theory with gauge group G_{SM} , provided it acts on all three factors at once; this is the only finite $\mathcal{N} = 2$ theory with this gauge group, while in the Standard Model the running couplings prevent S from acting as a map between theories of the same kind. The exact strong–weak dualities that a Seiberg–Witten construction provides on the $SU(2)$ factor sit at $\theta_2 = \pi$ with $g_2 g'_2 = 8\pi$; they are not the S-type elements S_Φ on which $\mathcal{M}$ and $\mathcal{M}_P$ are built, but their $SU(2)$ element is the one that enters the order-4 map of the $\mathbb{Z}_3$ quotient in Sec. III G. Second, line operators and 't Hooft anomaly matching do not fix which mirror generation an ordinary generation is mapped to; the action of $\mathcal{M}_P$ on the Yukawa couplings may fix it. Third, the origin of the n - n' mixing angle is open: it needs explicit breaking of $B - L$ between the sectors or effects on backgrounds that our analysis does not cover, and the sizes of the six-quark coefficients that $\mathcal{M}_P$ allows are not computed here; the relation of the small breaking $(v' - v)/v$ of Refs. [6, 7] to the couplings τ , τ' also remains to be studied. Gauging $\mathcal{M}_P$, which the absence of global symmetries in quantum gravity would require

for its protections to be exact, involves non-orientable backgrounds [89], whose reflection anomalies we have not studied. Fourth, the non-invertible self-duality defects found on special loci are characterized by their gauging data only; their topological field theories and fusion rules are not constructed. Finally, the four-dimensional mirror models with also self-dual gauge group per sector will be treated in the companion paper [30].

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Appendix A: Conventions

We use the conventions of Ref. [44] unless stated otherwise: natural units, the metric $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, and $\epsilon^{0123} = +1$, with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. A non-abelian factor has covariant derivative $D_\mu = \partial_\mu - igA_\mu^a t^a$ with $\text{tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$.

a. Theta term and coupling. The θ -term enters with a plus sign,

$$\mathcal{L} \supset + \frac{\theta g^2}{32\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad \tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}. \quad (\text{A1})$$

With these conventions $F_{\mu\nu} \tilde{F}^{\mu\nu} = -4\mathbf{E} \cdot \mathbf{B}$ is odd under parity and $F_{\mu\nu} F^{\mu\nu}$ is even, so that parity acts as $\tau \mapsto -\bar{\tau}$, and on the charges (n_m, n_e) as $(n_m, n_e) \mapsto (-n_m, n_e)$, for either sign of ϵ^{0123} or of the θ -term.

b. S-duality. We take

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tau \mapsto -\frac{1}{\tau}, \quad (\text{A2})$$

$$q = (n_m, n_e) \mapsto qS^{-1} = (n_e, -n_m),$$

acting on $(\tau, 1)$, and $T : \tau \mapsto \tau + 1$. In terms of electric and magnetic charges, S sends (e, m) to $(-m, e)$, and $S^2 = C$ is charge conjugation.

c. Abelian factors. For a $U(1)$ with integer charges and coupling e we use Eq. (A1) with $g^2 = 2e^2$:

$$\tau = \frac{\theta}{2\pi} + \frac{2\pi i}{e^2}, \quad \mathcal{L} \supset + \frac{\theta e^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (\text{A3})$$

so that S and T act on the integer charge lattice as in Eq. (A2). The hypercharge satisfies $Q = T_3 + Y$, with $Y(Q_L) = \frac{1}{6}$, $Y(L_L) = -\frac{1}{2}$ and $Y(H) = \frac{1}{2}$, and $D_\mu \supset -ig_Y Y B_\mu$; we write g_Y for the coupling g' of Ref. [44] and $g_1 = \sqrt{5/3}g_Y$ for the grand-unified normalization. The mirror hypercharge Y' is normalized in the same way,

$Y'(Q'_L) = \frac{1}{6}$. Since $Y \in \frac{1}{6}\mathbb{Z}$, the integer charge $y = 6Y$ defines the lattice normalization. The hypercharge θ -terms are those of Eq. (A1) with $g = g_Y$, plus a mixed term between the two sectors,

$$\mathcal{L} \supset + \frac{\theta_Y g_Y^2}{32\pi^2} B_{\mu\nu} \tilde{B}^{\mu\nu} + \frac{\theta_{\text{mix}} g_Y g'_Y}{16\pi^2} B_{\mu\nu} \tilde{B}'^{\mu\nu}, \quad (\text{A4})$$

and the hypercharge coupling is

$$\hat{\tau}_Y = \frac{3\theta_Y}{2\pi} + \frac{12\pi i}{g_Y^2}. \quad (\text{A5})$$

Kinetic mixing between the two hypercharge fields, with canonically normalized B and B' , is

$$\mathcal{L} \supset -\frac{\epsilon}{2} B_{\mu\nu} B'^{\mu\nu}, \quad |\epsilon| < 1, \quad (\text{A6})$$

and it combines with θ_{mix} into $\hat{\tau}_{\text{mix}} = 3\theta_{\text{mix}}/2\pi + 12\pi i\epsilon/(g_Y g'_Y)$.

d. Electric-magnetic duality on the Cartan coupling form. For a global form with cocharacter lattice $X_* \ni m$ and character lattice $X^* \ni e$, we write the Cartan (Coulomb-branch) Lagrangian as $\mathcal{L} = -\frac{1}{8\pi} \text{Im} \mathcal{T}(\mathcal{F}, \mathcal{F}) + \frac{1}{8\pi} \text{Re} \mathcal{T}(\mathcal{F}, \tilde{\mathcal{F}})$, with $\mathcal{F}$ the Cartan field strength and $\mathcal{T}$ a symmetric complex form. Given a root-datum isomorphism $\Phi : X^* \rightarrow X_*$ that maps roots to coroots (Sec. II A), S-duality acts as

$$\mathcal{T} \mapsto -\Phi^{-T} \mathcal{T}^{-1} \Phi^{-1}, \quad (\text{A7})$$

$$(m, e) \mapsto (\Phi e, -\Phi^{-T} m),$$

with central charge $Z = e + \mathcal{T}m \mapsto -\Phi^{-T} \mathcal{T}^{-1} Z$; it preserves the Dirac pairing $e_1 \cdot m_2 - e_2 \cdot m_1$ and $\text{Im} \mathcal{T} > 0$, squares to the identity on $\mathcal{T}$, and squares to C on the charges for symmetric Φ . For a single $U(1)$ with integer charges it reduces to Eq. (A2); for a simple factor, $\tau \mapsto -1/\tau$; and for the hypercharge of G_{SM} , $\hat{\tau}_Y \mapsto -1/\hat{\tau}_Y$.

Appendix B: When can S-duality hold for G_{SM} ?

In the main text S acts as a formal map on couplings, charge lattices and line operators. Here we collect what we have established about when it can be an exact equivalence for the gauge group G_{SM} : in $\mathcal{N} = 4$ and $\mathcal{N} = 2$ extensions, in the Standard Model itself, and for an exact $\mathcal{N} = 2$ duality on the $SU(2)$ factor alone. The supersymmetric settings suggested by the mirror models of Ref. [10] are discussed in Ref. [30]. Throughout, $\beta(g) = -b_0 g^3/16\pi^2$, so that $b_0 > 0$ means asymptotic freedom, and the abelian coefficients refer to the coupling g_Y of App. A.

1. $\mathcal{N} = 4$, $\mathcal{N} = 2$ and the running of the Standard Model

S-duality is expected to be exact in $\mathcal{N} = 4$ super-Yang-Mills theory [15–19], whose coupling does not run. With

gauge group G_{SM} the $\mathcal{N} = 4$ theory is one-loop conformal in every factor, $b_0(\mathfrak{su}_3) = b_0(\mathfrak{su}_2) = b_0(\mathfrak{u}_{1,Y}) = 0$, the $\mathfrak{u}_{1,Y}$ multiplet being free. On the line lattice $\Lambda_e \oplus \Lambda_m$ of Eq. (7), S built from any of the W -invariant self-duality maps of Sec. II A maps the lattice onto itself if it acts on all three factors at once or on none, and then $S^2 = C$, the Dirac pairing is preserved, and each Standard Model Wilson line goes to its magnetic dual of Table I. If S acts on a proper non-empty subset of the factors, the electric sublattice of the image projects onto the non-abelian labels (a_1, a_2, n) with index 2, 3 or 6, so the image is not the lattice of any theory with gauge group G_{SM} . The $\mathbb{Z}_6$ glue thus forces S to act on all factors simultaneously, and then the $\mathcal{N} = 4$ G_{SM} theories are mapped among themselves. That $\mathcal{N} = 4$ theories are finite to all orders and S -dual is taken from the literature [17, 36, 107]; our statement is the lattice one.

The non-supersymmetric Standard Model is far from this. With three generations and one Higgs doublet, $b_0 = (7, \frac{19}{6}, -\frac{41}{6})$ for $(\mathfrak{su}_3, \mathfrak{su}_2, \mathfrak{u}_{1,Y})$, or $-\frac{41}{10}$ for g_1 . For any $U(1)$ factor $b_0 = -\frac{2}{3} \sum_{\text{Weyl}} Y^2 - \frac{1}{3} \sum_{\text{scalars}} Y^2 < 0$ as soon as anything is charged, so an exactly marginal $\hat{\tau}_Y$ requires a free $U(1)_Y$. Gauge anomalies are no obstruction: all of them, including the $SU(2)$ Witten anomaly [108], vanish per generation for the Standard Model content and also for the two dual contents obtained from the self-duality maps ($y \rightarrow -y$, or color conjugation with $y \rightarrow y$); these are group-theory factors of triangle traces. Baryon and lepton number are an obstruction. Per generation, $B + L$ has the anomaly coefficients $(0, 1, -1)$ with $(SU(3)^2, SU(2)^2, U(1)_Y^2)$, while $B - L$ has none. A $B + L$ rotation by an angle α therefore shifts (θ_2, θ_Y) by $\pm 2\alpha n_g(1, -1)$, with n_g the number of generations (the sign is fixed by Eq. (37)), and only $\theta_2 + \theta_Y$ is invariant under it; the complete set of physical angles is given in Sec. III D. If S acted on τ_2 alone, it would map the physically equivalent couplings $\tau_2 + x$, $x \in \mathbb{R}$, at $\tau_2 = iy_2$ to

$$\text{Im } S(\tau_2 + x) = \frac{y_2}{x^2 + y_2^2}, \quad (\text{B1})$$

which depends on x , so S cannot act on τ_2 in the Standard Model as a map between theories of the same kind. Finally, in the electroweak vacuum the unbroken group is $H_{\text{em}} \cong U(3)$, and $\pi_1(H_{\text{em}}) \rightarrow \pi_1(G_{\text{SM}}) \cong \mathbb{Z}$ is an isomorphism, so that $\pi_2(G_{\text{SM}}/H_{\text{em}}) = 0$: the Standard Model has no smooth monopoles that could carry the dual charges of Table I.

Between these two extremes lie the $\mathcal{N} = 2$ extensions, in which $b_0 = 2C_2(G) - 2 \sum_{\text{hypers}} T(R)$, a half-hypermultiplet counting $-T(R)$, and $b_0(\mathfrak{u}_{1,Y}) = -2 \sum_{\text{hypers}} \dim R Y^2 \leq 0$. With one hypermultiplet for each Standard Model Weyl field of n_g generations and n_H hypermultiplets $(\mathbf{1}, \mathbf{2})_{1/2}$,

$$\begin{aligned} b_0(\mathfrak{su}_3) &= 6 - 4n_g, & b_0(\mathfrak{su}_2) &= 4 - 4n_g - n_H, \\ b_0(\mathfrak{u}_{1,Y}) &= -\frac{20}{3}n_g - n_H, \end{aligned} \quad (\text{B2})$$

so $\mathfrak{su}_3$ is never finite, $\mathfrak{su}_2$ is finite only for $(n_g, n_H) = (1, 0)$ and $(0, 4)$, and $\mathfrak{u}_{1,Y}$ only without charged hypermultiplets. More generally, exactly eight $SU(3) \times SU(2)$ hypermultiplet contents make both non-abelian factors finite, all free of the Witten anomaly. A finite $\mathfrak{u}_{1,Y}$ needs $Y = 0$ for every hypermultiplet, which in G_{SM} is allowed only for representations of triality zero and integer isospin; of the eight contents only $(\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{3})$, the adjoint, qualifies. Hence the only $\mathcal{N} = 2$ theory with gauge group G_{SM} that is finite in every factor is $\mathcal{N} = 4$, up to free neutral hypermultiplets.

2. A Seiberg–Witten tool on the $SU(2)$ factor

Of the factors of G_{SM} , only $SU(2)$ can be made conformal with Standard-Model-like hypermultiplets [Eq. (B2)]. $\mathcal{N} = 2$ $SU(2)$ with $N_f = 4$ flavors has an exact $SL(2, \mathbb{Z})$ duality acting together with the triality of its $Spin(8)$ flavor symmetry [109]. We use it as a tool: every chiral field is put into a hypermultiplet, so chirality is lost and the supercharges are neutral, and $SU(3)$ and $U(1)_Y$ are weakly gauged spectators. We write $\tau_{SW} = 2\tau_2$ for the coupling in the normalization of Ref. [109], and $q \mapsto q\gamma^{-1}$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$.

For one generation together with color-singlet doublets of hypercharge $\pm \frac{1}{2}$, the minimal hypermultiplet completion has

$$N_f = 3 + \max(n_+, n_-), \quad b_0(\mathfrak{su}_2) = 4 - N_f, \quad (\text{B3})$$

where $Q_L \oplus \bar{Q}_L$ supplies three flavors and $n_{\pm}$ counts the color-singlet doublets with $Y = \pm \frac{1}{2}$ (L_L counts in n_-). The tool is conformal, $N_f = 4$, if and only if $n_- = 1$ and $n_+ \leq 1$: with L_L and the Higgs doublet H it is conformal, while a second Higgs doublet of hypercharge $-\frac{1}{2}$, as in the pair (ϕ_u, ϕ_d) of Ref. [10], gives $N_f = 5$ and $b_0 = -1$. The $SU(2)^2 U(1)_R$ anomaly equals b_0 in every case; $b_0(\mathfrak{su}_3) = 2$ and $U(1)_Y$ is infrared free, so both stay non-conformal spectators. The kernel of the action on every completion is $\mathbb{Z}_6$, so the global form G_{SM} remains allowed, and in the conformal case the eight half-hypermultiplets restrict to $\mathbf{3}_{1/6} + \bar{\mathbf{3}}_{-1/6} + \mathbf{1}_{1/2} + \bar{\mathbf{1}}_{-1/2}$ under $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_Y$, the vector $\mathbf{8}_v$ of $Spin(8)$. Of the two spinors, one restricts in the same way, and the other to $\mathbf{3}_{-1/3} + \bar{\mathbf{3}}_{1/3} + 2 \times \mathbf{1}_0$, whose doublets are not in Λ_e ; we call it the bad spinor.

Which $SL(2, \mathbb{Z})$ elements are exact dualities of the tool with $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_Y$ gauged? We work in the frame whose line lattice is $\Lambda_m \oplus \Lambda_e$ and in which the $\mathbb{Z}_6$ kernel acts trivially on all states without a Witten shift. For the global forms $(SU(3) \times SU(2) \times U(1))/\mathbb{Z}_n$ with $n = 2, 6$ the $\mathbb{Z}_2$ part of the glue contains -1_2 , so the 't Hooft line of the fundamental coweight has $n_m = \frac{1}{2}$ and is mapped to $n_e = -b/2$; γ preserves the lattice if and only if b is even. In particular the S of the tool, S_{SW} , is excluded on G_{SM} for every action on the spectators. The monopole states with $n_m = 1$ carry the bad spinor in the class $(n_m, n_e) \equiv (1, 0)$, so the elements that preserve the gauged $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_Y$

are those that fix this class. For $n = 1, 2, 3$ and 6 the exact dualities form

$$\Gamma^0(2) = \left\{ \gamma \in SL(2, \mathbb{Z}) : b \equiv 0 \pmod{2} \right\}, \quad (\text{B4})$$

and every element of $\Gamma^0(2)$ has a physical completion by θ shifts of the spectators; on G_{SM} the element TST^{-1} needs, for example, $\Delta\theta_Y = \pm 2\pi$. Only ± 1 preserve the line $\theta_2 = 0$, so there is no strong-weak duality at $\theta_2 = 0$. $\Gamma^0(2)$ has one class of elliptic elements, represented by TST^{-1} , with fixed point $\tau_{SW} = 1 + i$, i.e. $\tau_2 = \frac{1+i}{2}$: $\theta_2 = \pi$ and $g_2^2 = 8\pi$. In $\tau_2 = \tau_{SW}/2$, $TST^{-1} : \tau_{SW} \mapsto (2 - \tau_{SW})/(1 - \tau_{SW})$ reads $\tau_2 \mapsto (1 - \tau_2)/(1 - 2\tau_2)$, which is the $SU(2)$ part of the order-4 map γ_3 of the $\mathbb{Z}_3$ quotient, Eq. (46), and $\Gamma^0(2)$ in τ_{SW} is the group $\Gamma_0(2)$ of Sec. III G in τ_2 : the two are conjugate under $\tau \mapsto 2\tau$.

The elements of the main text are not of this kind. All 24 root-datum isomorphisms of G_{SM} have $SU(2)$ block ± 1 , so every S_Φ acts on $SU(2)$ as $\tau_2 \rightarrow -1/\tau_2$, i.e. $\tau_{SW} \rightarrow -4/\tau_{SW}$, which is not in $SL(2, \mathbb{Z})$. Since every element $\mathcal{M}$, $\mathcal{M}_P$, $\tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}_P$ contains $S_\Phi \otimes S_{\Phi'}$, none is realized by the exact duality of the tool, for any global form; on $SU(3)$ and $U(1)_Y$ these elements were formal anyway. The strong-weak mirror elements that the tool does provide are built from γ_* , the element TST^{-1} on $SU(2)$ together with $\Delta\theta_Y = 2\pi$:

$$\begin{aligned} \mathcal{M}_{SW} &= \text{swap}(\gamma_* \otimes \gamma_*^{-1}), \\ \mathcal{M}_{P,SW} &= \text{swap}(P\gamma_* \otimes \gamma_*^{-1}P), \\ g_2 g_2' &= 8\pi \quad \text{at} \quad \theta_2 = \pi, \end{aligned} \quad (\text{B5})$$

with $\theta_2' = \pi$ for $\mathcal{M}_{SW}$ and $\theta_2' = -\pi$ for $\mathcal{M}_{P,SW}$ (equivalent to π up to a lattice-compatible shift). Both preserve the two-sector lattice and are involutions, so they are compatible with matter in the sense of Sec. III E; with equal couplings, $g_2 = g_2' = \sqrt{8\pi}$ at $\theta_2 = \theta_2' = \pi$, instead of $g_2 g_2' = 4\pi$ at $\theta_2 = 0$ for $\mathcal{M}_P$. The analogues of $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$, with a relative charge conjugation of the spectators, square to $C \otimes C$ and are not compatible with chiral matter. The tool is vector-like, so the $B + L$ obstruction of Eq. (B1) is untouched.

The same result holds for an $SU(2)$ that acts on the left-handed quarks only ($N_f = 4$ for one generation with a pair of Higgs-type doublets), as in one reading of the isospin group $SU_I(2)$ of Ref. [10]: the exact dualities that preserve the gauged spectators form $\Gamma^0(2)$ for every n , and the strong-weak point is $\theta = \pi$ with $gg' = 8\pi$. With three generations there is no conformal point, so neither S_{SW} nor TST^{-1} carries over.

3. Summary

S can be an exact equivalence for G_{SM} in the $\mathcal{N} = 4$ theory, where the $\mathbb{Z}_6$ glue forces it to act on all factors at once; among $\mathcal{N} = 2$ theories with gauge group G_{SM} this is the only one that is finite. In the Standard Model the

running couplings, the $B + L$ anomaly and the absence of smooth monopoles prevent S from acting on the couplings as a map between theories of the same kind. An $\mathcal{N} = 2$ Seiberg-Witten tool on the $SU(2)$ factor provides exact strong-weak elements, but they sit at $\theta_2 = \pi$ with $g_2 g_2' = 8\pi$ and are not the elements S_Φ of the main text. This is why S is used as a formal map in the main text.

Appendix C: The 2D charged $\mathcal{N} = (2, 2)$ model, T-duality and mirror symmetry

This appendix gives a two-dimensional model in which supercharges carry the charge of a dynamical gauge field, and shows how T-duality and the mirror elements of Sec. III F act on it. The ordinary sector is carried by the left-moving supercharge Q_+ and the mirror sector by the right-moving supercharge Q_- . The analysis is classical.

1. The model

We use light-cone coordinates $x^\pm = x^0 \pm x^1$, with $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$ and $\eta = \text{diag}(1, -1)$, and a $U(1)$ gauge field a_μ with $D_\mu = \partial_\mu - iqa_\mu$ on a field of charge q , field strength f_{+-} , coupling e and kinetic term $\mathcal{L}_a = -f_{\mu\nu}f^{\mu\nu}/4e^2 = 2f_{+-}^2/e^2$. The gauge field is not part of a supermultiplet: $\delta a_\mu = 0$. For a neutral complex scalar Z and fermions $\psi_\pm$ of charges q_L, q_R , the free model is supersymmetric with constant parameters as long as $a_\mu = 0$; after gauging, the variation of the Lagrangian is proportional to $q_L a_-$ and $q_R a_+$ and is not a total derivative. It becomes one when the parameters are dressed with Wilson lines. We count such parameters as global supersymmetry.

The general model of this kind has chiral multiplets (Z, ψ_+, ψ_-, F) with charges $(0, q_L, q_R, q_L + q_R)$ and twisted chiral multiplets $(\tilde{Z}, \chi_+, \chi_-, \tilde{G})$ with charges $(0, q_L, -q_R, q_L - q_R)$, and no superpotential. For one chiral multiplet with Kähler metric $g(Z, \bar{Z})$, $g_Z = \partial_Z g$, and $R = g_Z \bar{Z} - g_Z g_{\bar{Z}}/g$, the Lagrangian is

$$\begin{aligned} \mathcal{L} &= 2g(\partial_+ \bar{Z} \partial_- Z + \partial_- \bar{Z} \partial_+ Z) \\ &\quad + i\bar{\psi}_+(gD_- \psi_+ + g_Z \partial_- Z \psi_+) \\ &\quad + i\bar{\psi}_-(gD_+ \psi_- + g_Z \partial_+ Z \psi_-) \\ &\quad + \frac{1}{4}R\bar{\psi}_+ \psi_+ \bar{\psi}_- \psi_- + \mathcal{L}_a, \end{aligned} \quad (\text{C1})$$

and it is invariant, up to a total derivative, under

$$\begin{aligned} \delta Z &= -\frac{i}{4}(\bar{\epsilon}_+ \psi_+ + \bar{\epsilon}_- \psi_-), \\ \delta \psi_\pm &= \epsilon_\pm \partial_\pm Z - \frac{g_Z}{g} \delta Z \psi_\pm, \\ \epsilon_+ &= e^{iq_L u} \kappa_+(x^+), \quad \epsilon_- = e^{iq_R v} \kappa_-(x^-), \end{aligned} \quad (\text{C2})$$

where $a_- = \partial_- u$ and $a_+ = \partial_+ v$, which is always possible locally in two dimensions, and $\kappa_\pm$ are arbitrary profiles. The variation is a total derivative for every gauge field

and every metric, and in general invariance holds if and only if $D_- \epsilon_+ = D_+ \epsilon_- = 0$, which the dressed parameters solve; within the ansatz of transformations linear in the fields all coefficients are unique. For a flat target, $g = 1$, with $+4\bar{F}F$ in the Lagrangian, the off-shell transformations are $\delta\psi_+ = \epsilon_+ \partial_+ Z + \bar{\epsilon}_- F$, $\delta\psi_- = \epsilon_- \partial_- Z - \bar{\epsilon}_+ F$ and $\delta F = \frac{i}{4}(\epsilon_+ D_+ \psi_- - \epsilon_- D_- \psi_+)$; the twisted chiral multiplet transforms with $\epsilon_- \leftrightarrow \bar{\epsilon}_-$ and the same coefficients. The algebra closes off shell,

$$[\delta_1, \delta_2]\Upsilon = \xi^+ D_+ \Upsilon + \xi^- D_- \Upsilon + i(-\ell_\Upsilon \xi^+ \partial_+ \varpi + r_\Upsilon \xi^- \partial_- \varpi)\Upsilon, \quad (\text{C3})$$

for every field Υ , with $\xi^\pm = \frac{i}{4}(\kappa_{1\pm} \bar{\kappa}_{2\pm} + \bar{\kappa}_{1\pm} \kappa_{2\pm})$, $\varpi = u - v$, so that $\partial_+ \partial_- \varpi = f_{+-}$, and $\ell_\Upsilon, r_\Upsilon$ the charges of Υ carried by its u and its v dressing; since a_μ does not transform, the extra rotation is not a gauge transformation. The dressed supersymmetry commutes with every gauge transformation, which shifts u and v alike, and for Wilson lines anchored at a base point Q_+ and Q_- carry the charges q_L and q_R of the global $U(1)$. Classically the gauge field decouples: with $\psi_+ = e^{iq_L u} \hat{\psi}_+$, $\psi_- = e^{iq_R v} \hat{\psi}_-$ and $F = e^{i(q_L u + q_R v)} \hat{F}$, the Lagrangian is the ungauged one in $(Z, \hat{\psi}_\pm, \hat{F})$ plus $\mathcal{L}_a$, and the dressed supersymmetry is the ordinary one with constant parameters. The quantum dynamics of the gauged model is not studied here.

2. Interactions and anomalies

Two kinds of interaction are excluded. A scalar of charge $t \neq 0$ gives an obstruction proportional to $t f_{+-}$. A superpotential W , which the charges allow only for $q_R = -q_L$, or a twisted superpotential, allowed only for $q_R = q_L$, gives an obstruction proportional to $q_L \partial_- \varpi \bar{W}''$; in the dressed frame the Yukawa term is $e^{iq_L \varpi} \frac{i}{4} W'' \hat{\psi}_+ \hat{\psi}_-$, so the gauge field stays coupled. Both obstructions vanish for $t = 0$, for $W = 0$, and on flat backgrounds, $f_{+-} = 0$, with covariantly constant parameters, and for completely general parameters a jet analysis at a point with generic a_μ forces all four parameters to vanish there. With the gauge field outside any multiplet, the admissible interactions are therefore those of Kähler sigma models with $W = \bar{W} = 0$ and neutral scalars, the class of Eq. (C1); charged gauginos and the Yukawa couplings of gauged linear sigma models [110] are not covered.

For N_c chiral multiplets on a Kähler target $\mathcal{X}$ and N_t twisted chiral multiplets on $\tilde{\mathcal{X}}$, with fermion charges (q_L, q_R) and $(q_L, -q_R)$, the gauge anomaly polynomial is

$$I_4 = (q_L - q_R) c_1(L_a) c_1(\mathcal{X}) + (q_L + q_R) c_1(L_a) c_1(\tilde{\mathcal{X}}) + \frac{1}{2}(N_c + N_t)(q_L^2 - q_R^2) c_1(L_a)^2, \quad (\text{C4})$$

with L_a the line bundle of the gauge field, and $c_L = c_R$. The gauge field can be gauged consistently if and only if $q_R = \pm q_L$, with $c_1(\tilde{\mathcal{X}}) = 0$ on the vector branch $q_R = q_L$

and $c_1(\mathcal{X}) = 0$ on the axial branch $q_R = -q_L$; flat targets allow both. The axial case agrees with the known anomaly of the axial R-symmetry, proportional to c_1 of the target [106, 110]. The other R-symmetry combination has the mixed anomaly $2q_L(N_c + N_t)$ with the gauge field. With several gauge fields acting on Q_+ and Q_- through charge vectors q and q' , the anomaly matrix $(N_c + N_t)(qq^T - q'q'^T)$ vanishes if and only if $q' = \pm q$, so at most one $U(1)$ combination can act on the supercharges. In particular an ordinary $U(1)$ acting only on Q_+ and a mirror $U(1)'$ acting only on Q_- cannot both be gauged. The single anomaly-free gauge field couples through a_- to the left-moving current of the ordinary supercharge Q_+ and through a_+ to the right-moving current of the mirror supercharge Q_- .

3. T-duality and mirror symmetry

On a flat torus the model has the moduli of the compact bosons of Sec. IIIF. With $Z = X^1 + \tau X^2$, $X^i \sim X^i + 2\pi$, and the constant metric $g = \rho_2/(4\pi\tau_2)$, where $\tau = \tau_1 + i\tau_2$ and $\rho = \rho_1 + i\rho_2$ are the complex and Kähler moduli of the torus, the bosonic part of the chiral model is the two-boson model of Eq. (44) with $G = (\rho_2/\tau_2) \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$ and $b = \rho_1$. The B -field term is topological, hence invariant under the dressed supersymmetry, so the exactly marginal couplings are (τ, ρ) ; the gauge coupling and the charges q_L, q_R do not enter the background. T-duality in its Buscher form (see Ref. [46]), with a_μ, e, q_L and q_R as spectators, acts along X^1 or along X^2 as

$$\begin{aligned} X^1 : (\tau, \rho) &\mapsto (-\bar{\rho}, -\bar{\tau}), \\ X^2 : (\tau, \rho) &\mapsto \left(\frac{1}{\bar{\rho}}, \frac{1}{\bar{\tau}}\right). \end{aligned} \quad (\text{C5})$$

The map of the left movers is holomorphic and that of the right movers antiholomorphic, so the dual is the twisted chiral model with the same labels: $\chi_+ = c_+ \psi_+$, of charge q_L , and $\chi_- = c_- \bar{\psi}_-$, of charge $-q_R$, with $g'|c_\pm|^2 = g$. For one flat multiplet this is the chiral model with charges $(q_L, -q_R)$: T-duality exchanges the two anomaly-free branches, the vector branch $q_R = q_L$ and the axial branch $q_R = -q_L$, and preserves the coefficient $q_L^2 - q_R^2$ of $c_1(L_a)^2$ in Eq. (C4). This exchange of chiral and twisted chiral multiplets is the mirror symmetry of the $\mathcal{N} = (2, 2)$ sigma model on a torus, which is T-duality [106].

The charged supercharges go along. On shell, the dressed supersymmetry of Eq. (C2) is mapped to the dressed supersymmetry of the dual with the identity on the parameters: the charges q_L of Q_+ and q_R of Q_- and the Wilson-line dressings are unchanged, and in the chiral labeling the map is the automorphism $Q_- \leftrightarrow \bar{Q}_-$ of the algebra, which exchanges its vector and axial forms. Among all signed permutations of $(\epsilon_+, \bar{\epsilon}_+, \epsilon_-, \bar{\epsilon}_-)$, compared with both supersymmetries of the dual, this is the

TABLE VIII. The two-dimensional mirror elements on the gauged charged $\mathcal{N} = (2, 2)$ model on a torus: fixed locus in the moduli (τ, ρ) , action on the supercharges, and the charges for which the element is a symmetry without or with C_a ($a_\mu \rightarrow -a_\mu$).

Element	Fixed locus	Supercharges	No C_a	With C_a
M_{2D}	$\tau_1 = 0, \rho = 1$	$Q_\pm \leftrightarrow Q_\pm$	none	all
$\tilde{M}_{2D}$	$\rho = i$	fixed	all	none
$M_{P,2D}$	$\tau_1 = 0, \rho = i$	$Q_+ \leftrightarrow \bar{Q}_-$	$q_R = -q_L$	$q_R = q_L$
$\tilde{M}_{P,2D}$	$ \rho = 1$	$Q_+ \leftrightarrow Q_-$	$q_R = q_L$	$q_R = -q_L$

only match, up to relabeling. The dressing is needed for the duality itself: the dual variation $\delta Z'$ is local if and only if $\partial_+ \partial_- \delta Z = 0$ on shell, which holds because the dressed combinations $\bar{\epsilon}_\pm \psi_\pm$ are chiral; with undressed parameters and $a_\mu \neq 0$ it fails, by terms proportional to $q_L a_-$ and $q_R a_+$.

4. The mirror elements

Here the word “sector” changes meaning. In Sec. III F the two sectors are the two bosons X^1 and X^2 , one per sector; in this appendix the ordinary and mirror sectors are carried by the supercharges Q_+ and Q_- , and X^1, X^2 are the two real coordinates of the field $Z = X^1 + \tau X^2$ of a single multiplet. The elements M_{2D} = swap $(T \otimes T)$, $M_{P,2D}$ = swap $\Omega(T \otimes T)$ and their lifts $\tilde{M}_{2D} = M_{2D}(1 \otimes C)$, $\tilde{M}_{P,2D} = M_{P,2D}(1 \otimes C)$ of Sec. III F act on the two bosons X^1, X^2 . We write C_a for $a_\mu \rightarrow -a_\mu$ and let the fermions transform like $\partial_\pm X$. A map is then a symmetry of the gauged model at its fixed locus, which is unchanged from the bosonic one, if and only if it maps the charges (q_L, q_R) to $s(q_L, q_R)$, with $s = -1$ meaning that it is combined with C_a . The result is in Table VIII.

The ordinary and mirror supercharges are thus exchanged exactly by the elements of the worldsheet-parity coset, $M_{P,2D}$ and $\tilde{M}_{P,2D}$, and these need $|q_R| = |q_L|$, which is the anomaly-free condition of Eq. (C4). M_{2D} and $\tilde{M}_{2D}$ exchange the two bosons but not the supersymmetry sectors. Of the other elements, swap and C on the second boson need C_a , while $T \otimes T$ and $C \otimes C$ do not, and Ω at $\rho_1 = 0$ behaves as $\tilde{M}_{P,2D}$. The group $\langle M_{2D}, M_{P,2D} \rangle \cong D_4$ acts on $(\epsilon_+, \bar{\epsilon}_+, \epsilon_-, \bar{\epsilon}_-)$ through $\mathbb{Z}_2 \times \mathbb{Z}_2$; its center $C \otimes C$ acts trivially on the supercharges and as $X \rightarrow -X, \psi \rightarrow -\psi$ on the fields. A θ -term $\theta_a \epsilon^{\mu\nu} f_{\mu\nu}$ is odd under worldsheet parity and under C_a and even under $O(2, 2; \mathbb{Z})$, so the elements that use exactly one of the two, such as M_{2D} , or $M_{P,2D}$ on the axial branch, require $\theta_a \equiv -\theta_a$ modulo its period.

Since T-duality exchanges the vector and axial branches [Eq. (C5)], whether a conjugation of the supercharges accompanies worldsheet parity depends on the T-duality frame, while worldsheet parity itself is needed in every frame. Under the lattice dictionary of Eq. (45), $M_{P,2D} = \mathcal{M}(C \otimes C)$ and $\tilde{M}_{P,2D} = \tilde{\mathcal{M}}(C \otimes C)$ preserve

orientation in four dimensions, so the exchange of chiralities in two dimensions maps to elements that keep the helicity in four. On the glued Standard Model lattice with chiral matter, $\mathcal{M}(C \otimes C)$ preserves no content in either S-dual frame, while $\tilde{\mathcal{M}}(C \otimes C)$ does, with mirror labels in frame D and copy labels in frame D' (Sec. III E). The difference between the two dimensions is that $C \otimes C$ is a symmetry of the two-dimensional model but not of chiral matter in four dimensions.

Appendix D: Proof of the self-duality criteria

We prove Eqs. (14) and (16). Let $G = S(GL_{n_1} \times \cdots \times GL_{n_k})$, $N = \sum_i n_i$, with the lattices of Eq. (1) on $\mathbb{Z}^N$: $X^* = \mathbb{Z}^N / \mathbb{Z}(1, \dots, 1)$ and $X_* = \{\lambda \in \mathbb{Z}^N : \sum_a \lambda_a = 0\}$, paired by the dot product. The roots and the coroots are the vectors $u_a - u_b$ with $a \neq b$ in the same block. For $x \in \mathbb{Z}^N$ we write $x^{(i)}$ for its restriction to block i and $S_i(x)$ for its sum over block i , and set $S = (S_1, \dots, S_k)$, $n = (n_1, \dots, n_k)$, $d = \gcd(n)$ and $m_j = \gcd\{n_i : i \neq j\}$, with $\gcd \emptyset = 0$. An isomorphism $\mathcal{R} \rightarrow \mathcal{R}^\vee$ is a lattice isomorphism $f : X^* \rightarrow X_*$ that maps the roots onto the coroots and whose transpose maps β to $\alpha^\vee$ whenever $f(\alpha) = \beta^\vee$ (Sec. II A). By the isomorphism theorem for connected reductive groups [28, 29], G (or its compact form) is self-dual if and only if such an f exists. We also use two facts about root systems: an isomorphism maps irreducible components onto components of the same type, and the automorphism group of A_{r-1} is $S_r \times \{\pm 1\}$ for $r \geq 3$ and $\{\pm 1\}$ for $r = 2$.

a. Step 1: the quotients by the roots. The map $x \mapsto S(x)$ identifies $X^*/\mathbb{Z}\Phi$ with $\mathbb{Z}^k/\mathbb{Z}n \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}_d$, and $X_*/\mathbb{Z}\Phi^\vee$ with $\mathbb{Z}_0^k = \{T \in \mathbb{Z}^k : \sum_i T_i = 0\} \cong \mathbb{Z}^{k-1}$. Since f maps $\mathbb{Z}\Phi$ onto $\mathbb{Z}\Phi^\vee$, it induces an isomorphism of these quotients, so $d = 1$ is necessary. We assume $d = 1$ from now on.

b. Step 2: the form of f . By the two facts, f maps the roots of block i to the coroots of a block $\pi(i)$ of the same size, as $\alpha \mapsto s_i \sigma_i(\alpha)$, with $s_i = \pm 1$ and σ_i a bijection from block i to block $\pi(i)$ acting on the indices (for $n_i = 2$ we take $s_i = 1$; blocks of size 1 have no roots and are paired arbitrarily). The transpose condition gives $\langle \sigma_i(\alpha), f(x) \rangle = s_i \langle x, \alpha^\vee \rangle$ for every root α of block i , which fixes the differences of the coordinates of $f(x)$ in block $\pi(i)$:

$$f(x)^{(\pi(i))} = s_i x^{(i)} \circ \sigma_i^{-1} + c_i(x)(1, \dots, 1). \quad (\text{D1})$$

The offset c_i is an integer linear form that vanishes on the roots, since $f(\alpha)$ is a coroot, so it depends only on the block sums, $c_i(x) = (AS(x))_i$ with $A \in \mathbb{Z}^{k \times k}$. That f is defined on X^* , $f(1, \dots, 1) = 0$, gives $An = -s$, and that it lands in X_* gives $n^T A = -s^T$. Conversely, any such data satisfy both root conditions.

c. Step 3: reduction to the glue. Since f maps $\mathbb{Z}\Phi$ bijectively onto $\mathbb{Z}\Phi^\vee$, it is bijective if and only if the induced map of the quotients of Step 1 is. By Eq. (D1)

that map sends S to T with $T_{\pi(i)} = s_i S_i + n_i (AS)_i$. After relabeling the blocks by π , which preserves their sizes, $\mathcal{R} \cong \mathcal{R}^\vee$ if and only if there are signs s_i and an isomorphism $\varphi: \mathbb{Z}^k/\mathbb{Z}n \rightarrow \mathbb{Z}_0^k$ with

$$\varphi(S)_i \equiv s_i S_i \pmod{n_i}, \quad i = 1, \dots, k; \quad (\text{D2})$$

given such a φ , $n_i (AS)_i = \varphi(S)_i - s_i S_i$ defines A , and Eq. (D1) defines f . The quotients of Step 1 record how the blocks are glued together in X^* and in X_* , and Eq. (D2) states that f must carry one gluing onto the other. It constrains φ only modulo $L = \text{lcm}(n)$. We orient $\mathbb{Z}^k/\mathbb{Z}n$ and $\mathbb{Z}_0^k$ by the forms $\det[n, \cdot]$ and $\det[e_1, \cdot]$; then φ is an isomorphism if and only if it multiplies the orientations by a sign $\eta = \pm 1$.

d. Step 4: necessity. Fix j . The classes of the unit vectors e_l , $l \neq j$, form a basis of a sublattice of $\mathbb{Z}^k/\mathbb{Z}n$ of index n_j , and the $e_i - e_j$, $i \neq j$, form a basis of $\mathbb{Z}_0^k$ in which T has the coordinates T_i , $i \neq j$. In these bases the matrix of φ has determinant ηn_j , and by Eq. (D2) its entries satisfy $\varphi(e_l)_i \equiv s_i \delta_{il}$ modulo n_i , hence modulo m_j , which divides every n_i with $i \neq j$. Therefore

$$n_j \equiv \eta \prod_{i \neq j} s_i \pmod{m_j} \quad \text{for every } j, \quad (\text{D3})$$

with one common choice of (s, η) . In particular $n_j \equiv \pm 1$ modulo m_j for every j , which is Eq. (16). This condition also implies $d = 1$: a prime dividing every n_i divides m_j and n_j , and then $n_j \not\equiv \pm 1$.

e. Step 5: sufficiency. Suppose $n_j \equiv t_j \pmod{m_j}$ with $t_j = \pm 1$ for every j . Signs with $\eta \prod_{i \neq j} s_i = t_j$ for all j always exist: with $\sigma = \prod_i s_i$ the conditions read $s_j = \eta \sigma t_j$, which are solved by $\sigma = \prod_i t_i$ and any η for even k , and by $\eta = \prod_i t_i$, $\sigma = 1$ for odd k . It then suffices to construct, modulo each prime power p^a that exactly divides L , a matrix satisfying Eq. (D2) with determinant η : the local solutions combine by the Chinese remainder theorem into a solution modulo L , which lifts to an integer matrix with determinant η because $SL_{k-1}(\mathbb{Z}) \rightarrow SL_{k-1}(\mathbb{Z}/L)$ is surjective. At p , choose j with $p \nmid n_j$ and use the bases of Step 4, which are bases over the p -adic integers; the required determinant is the unit ηn_j , and the rows $i \neq j$ are fixed modulo $p^{v_p(n_i)}$ to $s_i e_i^T$. We start from the diagonal matrix $\text{diag}(s_i)$ and

multiply one row by a unit. If $p \nmid n_i$ for some $i \neq j$, row i is unconstrained and carries the determinant. Otherwise p divides m_j , the determinant of the diagonal matrix is $\prod_{i \neq j} s_i \equiv \eta n_j$ modulo $p^{v_p(m_j)}$ by Eq. (D3), and the remaining factor, which is 1 modulo $p^{v_p(m_j)}$, is put on a row with $v_p(n_i) = v_p(m_j)$. Expressed in fixed $\mathbb{Z}$ -bases of $\mathbb{Z}^k/\mathbb{Z}n$ and $\mathbb{Z}_0^k$ that carry the orientations of Step 3, each local solution has determinant η , since the bases of Step 4 change the determinant by the factor n_j , a unit at p ; the Chinese remainder theorem therefore applies in these bases. This proves Eq. (16). The sign patterns s that occur are exactly those of Eq. (D3); $s_i = -1$ means that the factor $U(n_i)$ is charge conjugated.

f. Two blocks. For $k = 2$ and $(n_1, n_2) = (m, n)$, $m_1 = n$ and $m_2 = m$, and Eq. (D3) reads

$$n \equiv \eta s_1 \pmod{m}, \quad m \equiv \eta s_2 \pmod{n}, \quad (\text{D4})$$

where η is now the sign with which f maps the central character $r^* = [(1^m, 0^n)]$ to the primitive central cocharacter $r_* = (n 1^m, -m 1^n)$. The same conditions follow directly from the glue. Up to W , which acts simply transitively on the bases of each block and trivially on weights modulo roots, f is one of the eight maps $F(s_1, s_2, \eta)$ with $F\alpha = s_i \alpha^\vee$ on block i and $F r^* = \eta r_*$. In $(P_m/Q_m) \times (P_n/Q_n) \times (\mathbb{Q} r^*/\mathbb{Z} r^*)$, with P/Q the weights modulo the roots of each block, X^* is generated over $\mathbb{Z}\Phi \oplus \mathbb{Z} r^*$ by $[u_1] \mapsto (1, 0, \frac{1}{m})$ and $[u_{m+1}] \mapsto (0, 1, -\frac{1}{n})$, and X_* over $\mathbb{Z}\Phi^\vee \oplus \mathbb{Z} r_*$ by $u_1 - u_N \mapsto (1, -1, \frac{1}{mn})$, in units of r_* ; both glue groups have order mn . $F(s_1, s_2, \eta)$ maps the first glue group onto the second if and only if Eq. (D4) holds. For $1 < m < n$, $m \equiv \pm 1$ modulo n forces $m = n - 1$, and then $n \equiv 1$ modulo m ; for $m = n$ coprimality requires $m = n = 1$. This gives the closed form of Eq. (14). For $n = m + 1$ and for $m = 1$ the isomorphisms are also realized at the group level by explicit maps, Eq. (15) for the first. When $\mathcal{R} \cong \mathcal{R}^\vee$, the isomorphisms form a torsor under $\text{Aut } \mathcal{R}$, so $|\text{Isom}(\mathcal{R}, \mathcal{R}^\vee)| = |\text{Aut } \mathcal{R}|$. For $(m, n) = (2, 3)$, $s_1 = -1$ is a Weyl reflection of $SU(2)$, the first condition of Eq. (D4) is empty, and the second gives $\eta s_2 = -1$; with $s_1 = \epsilon_2$ and $s_2 = \epsilon_3$, and with η the sign on the central directions, this is the count of Sec. II A.

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