# Mirror Duality in the SMM4 and SMM4b Models

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We study the mirror duality of the companion paper, built on the Langlands self-duality of the Standard Model gauge group $G_{\rm SM}$, in the four-dimensional mirror-matter models SMM4 and SMM4b with gauge group $G_{\rm SM} \times G'_{\rm SM}$. In SMM4, reading the $SU(2)$ factors of the left-handed leptons and quarks as two halves of one gauge $SU(2)$ makes each sector exactly $G_{\rm SM}$, anomaly free and self-dual without added matter; with shared neutrinos, only the mirror parity swap $\circ\,(P \otimes P)$ and its CP partner swap $\circ\,(C \otimes C)$ preserve the content, and elementary quarks cannot be shared. In SMM4b with shared neutrinos the bosonic and fermionic degrees of freedom balance, at 90 per sector, as a count of broken generators and not as a supersymmetry. Sharing neutral quark condensates as well identifies only vacuum values, and it forces equal vacuum values on the shared quark type. A flow from SMM4 to SMM4b through four-fermion interactions realizes the symmetry-breaking pattern of SMM4b but not the Standard Model spectrum, and its baryon-number-violating couplings must be strongly suppressed; the strong–weak part of the duality, a formal map from the outset, is lost at condensation, and the flow contains an axion of the Weinberg–Wilczek type. Spontaneous mirror breaking at the size needed for neutron–mirror-neutron oscillations requires tuning and a cosmological history that selects a single minimum in each sector, and in the two-doublet model the tuned vacuum puts the mirror sector at $\bar\theta' = \bar\theta + \pi$, far outside the oscillation window, unless $\bar\theta$ is relaxed in each sector. In an $\mathcal{N} = 2$ model with a single $G_{\rm SM}$ the mirror fermions are the superpartners of the ordinary ones, but its two-sector form cannot be exchange symmetric for an odd number of generations.

## I. INTRODUCTION

In mirror-matter theories each ordinary particle has a mirror partner, and the gauge group is a product $G \times G'$ of an ordinary and a mirror sector that share spacetime and gravity, related by an exchange of the sectors combined with parity [1–5]. In the mirror-matter theory of Refs. [6–10] both factors are the Standard Model gauge group, $G = G' = G_{\rm SM}$, and the two sectors are connected, besides gravity, by oscillations of neutral hadrons into their mirror partners, $n$–$n'$ and $K^0$–$K^{0\prime}$ above all. Reference [10] realizes this theory in two four-dimensional models. In SMM4 each sector carries one generation, which is read as the exterior algebra $\Lambda(\mathbb{C}^5)$ of a split tangent space $\mathbb{C}^5 = \mathbb{C}^2 \oplus \mathbb{C}^3$ [11, 12]; the left-handed leptons and quarks are doublets of two $SU(2)$ factors, $SU_L(2)$ and $SU_I(2)$, and the neutrinos are shared between the sectors. In SMM4b each sector carries three generations, there is no elementary Higgs field, and quark condensates break the electroweak symmetry; its bosonic and fermionic degrees of freedom balance, a property that Ref. [10] calls pseudo-supersymmetry.

The companion paper [13] (Paper I) relates the mirror map to electric–magnetic duality. The Standard Model gauge group $G_{\rm SM} = S(U(2) \times U(3))$ is Langlands self-dual, i.e. isomorphic to the group whose electric charges are its magnetic charges, so a strong–weak ($S$) map takes a $G_{\rm SM}$ theory to a $G_{\rm SM}$ theory. On the product $G_{\rm SM} \times G'_{\rm SM}$ this defines mirror elements, among them $\mathcal{M}_P = \text{swap} \circ P \circ (S \otimes S)$, which exchanges the sectors and reverses the orientation of spacetime; in the $S$-dual frame of the mirror sector $\mathcal{M}_P$ is the standard mirror parity swap$\circ(P\otimes P)$. As there, $S$ is a formal map on couplings, charge lattices and line operators, since S-duality is expected to be exact only in $\mathcal{N} = 4$ super-Yang–Mills theory [14, 15]. Paper I works in a model-independent setting, with one Standard Model generation per sector and an elementary Higgs doublet, and it classifies the operators that couple the sectors (portals) up to dimension six under the mirror elements. In the present paper the sectors are coupled only through gravity and the shared fields: we exclude portal operators throughout, so that classification is not used here.

Here we ask how this mirror duality works in SMM4 and SMM4b, where the questions become model dependent. How must $SU_I(2)$ be read so that each sector has a self-dual and anomaly-free gauge group? What may the two sectors share without violating gauge invariance, and do the mirror maps respect what is shared? Can SMM4 flow dynamically to SMM4b, and what is left of the duality below the scale where the quarks condense? Our main scenario is the neutrino-sharing one, in which only the neutrinos are shared, as in Ref. [10]. We also test an alternative, the condensate-sharing scenario, in which neutral quark condensates are shared as well; since a composite shared between the sectors couples to the quarks of both, such sharing is a candidate origin of the neutral-hadron oscillations, which motivates it. In this scenario we postulate the counts $n_f = 32 - 2 - 6 = 24$ per sector in SMM4, and $n_b = n_f = 72$ per sector in SMM4b, with the flavor symmetry of the left-handed quarks broken. These counts are introduced here; Ref. [10] states

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only $n_b = n_f = 30$ for one generation of SMM4 and $n_b = n_f = 90$ per sector for SMM4b.

Our main results are the following. If $SU_L(2)$ and $SU_I(2)$ are the lepton and quark halves of one gauge $SU(2)$, each sector is exactly $G_{\rm SM}$, anomaly free with no added matter and Langlands self-dual, for one and for three generations. With the neutrinos shared, the two-sector theory is self-dual and anomaly free, and of the mirror elements exactly $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ preserve its content, in their classical frame forms swap $\circ\,(P \otimes P)$ and swap $\circ\,(C \otimes C)$. Elementary quarks cannot be shared in a gauge-invariant way, and the duality neither requires nor allows it. The alternative readings in which $SU_I(2)$ and a $U_{BL}$ are gauged separately, with gauge group $G_{\rm SM} \times U(2)$ per sector, are self-dual only for some charge classes and are anomalous; each needs added matter.

In SMM4b with shared neutrinos, the chiral flavor group $U_f(6)_L \times U_f(6)_R$ of the six quarks is broken to give 63 pseudo-Goldstone bosons, and $n_b = n_f = 90$ per sector as a count of degrees of freedom, i.e. of broken generators, not of light states; the helicity content is that of $\mathcal{N} = 4$ super-Yang–Mills theory, but the representations are not, so no supersymmetry acts. We realize the flow from SMM4 to SMM4b with four-fermion interactions in each sector, which make the quark bilinears condense as in the Nambu–Jona-Lasinio model [16, 17], and with an $S_3$-symmetric input that lifts the degeneracy of the generations. 't Hooft anomalies match along the flow, and the balance of bosons and fermions appears only in the infrared, where the composite bosons supply it; in these dynamical chains most of the 63 broken generators are heavy. The chains are a dynamical existence proof for the symmetry-breaking pattern of SMM4b, not a realistic model. Their neutrinos are massless: a Dirac mass would be a portal, and a Majorana mass needs a per-sector $\Delta L = 2$ operator, absent at the dimension considered and incompatible with the $B - L$ of each sector, which is exact up to its gravitational anomaly. Their charged leptons are massless unless the discrete symmetry that protects the vacua is modified, and all six quarks have masses of the same order. Their baryon-number-violating four-fermion couplings must be suppressed far below the size of the condensing ones to respect the proton lifetime; this suppression leaves the vacuum structure and the flat direction of the chains unchanged. In the condensate-sharing alternative, quark composites can be shared gauge invariantly only as color-singlet scalars after electroweak breaking or as vector currents. Without portals the sharing identifies vacuum values, not fields, and the balance $n_b = n_f = 72$ holds only as a bookkeeping of degrees of freedom. Sharing a vacuum value forces $\varepsilon = 0$, i.e. equal vacuum values in the two sectors, on the shared quark type, so that the $n$–$n'$ window of Ref. [9] is out of reach when both quark types are shared.

The mirror maps act consistently on the condensates, the pseudo-Goldstone bosons and the vacua, but their strong–weak part does not survive the condensation: below that scale only the classical frame forms of the mirror elements remain, with equal couplings in the two sectors. The duality operator with $\psi_L \to -\psi'_R$, $\psi_R \to \psi'_L$ and $H \to -H'$ suggested by Ref. [10] is $\mathcal{M}_P$ up to a gauge transformation. With exact mirror symmetry the two sectors can still settle in different local minima, but the resulting asymmetry of the vacuum values is large unless a parameter is tuned to give the asymmetry $\varepsilon = (v' - v)/v \sim 10^{-14}$ that the $n$–$n'$ oscillations of Ref. [9] require. Even then a cosmological history must select a single minimum in each sector, since domain walls between the minima would otherwise remove the asymmetry or, if stable, conflict with cosmology. In SMM4 with two elementary Higgs doublets per sector (the two-doublet model) the minimum that gives the asymmetry puts the mirror sector at $\bar{\theta}' = \bar{\theta} + \pi$, with a very different hadron spectrum, unless $\bar{\theta}$ is relaxed in each sector separately. The four-fermion chains have an exact flat direction that relaxes the strong CP angle in each sector, at the price of an axion of the Weinberg–Wilczek type, with decay constant of the order of the electroweak scale; we state this as an open problem. Finally, in an $\mathcal{N} = 2$ theory with a single $G_{\rm SM}$, the hypermultiplet partners of the Standard Model fermions are exactly their mirror fermions, and the theory has a parity that is $\mathcal{M}_P$ without its $S$ part. Its unfolding into two sectors, $G_{\rm SM} \times G'_{\rm SM}$, cannot be symmetric under the exchange of the sectors for an odd number of generations; $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ are then dualities between two quivers with the same diagonal theory.

With $S$ formal, the mirror symmetry that acts in the infrared of these models is the classical mirror parity swap $\circ\,(P \otimes P)$ or its CP partner swap $\circ\,(C \otimes C)$, i.e. the exchange symmetry of the mirror-matter literature since Refs. [1–3]. As in Paper I, the results that do not depend on the frame in which the mirror sector is described, nor on whether $S$ is an exact equivalence, are the following. (i) The matter-compatibility rule: only elements of the $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ types preserve the two-sector content, with shared neutrinos and in the $\mathcal{N} = 2$ quivers. (ii) Elementary quarks cannot be shared in a gauge-invariant way. (iii) The descent criterion: a mirror element of the ultraviolet induces one in the infrared if and only if it preserves the charges along the light directions of the Higgs phase, which every $S$-type element fails in the original frame. (iv) 't Hooft anomaly matching along the flow. (v) The no-go for an exchange-symmetric $\mathcal{N} = 2$ quiver with an odd number of generations.

The paper is organized as follows. Section II recalls what we take from Paper I and describes one generation as $\Lambda(\mathbb{C}^5)$. Section III treats SMM4 in the one-SU(2) reading: its gauge group, the sharing of neutrinos, quarks and quark composites, its six-dimensional lift and the status of $S$. Section IV gives the content and counting of SMM4b, Sec. V the flow from SMM4 to SMM4b, and Sec. VI the mirror maps in SMM4b, their descent, and spontaneous mirror breaking. Section VII compares the readings with gauge group $G_{\rm SM} \times U(2)$, and Sec. VIII summarizes what changes with respect to Paper I and

concludes. The appendices give the conventions and the abelian mixing of the $G_{\rm SM}\times U(2)$ readings (App. A), anomaly coefficients and searches (App. B), the descent through the link-field breaking (App. C), supersymmetry in the mirror models (App. D), the $\mathcal{N}=2$ model (App. E), and the mean-field model, the samples and the vacua of the chains and of the two-doublet model (App. F).

## II. SETTING: ONE GENERATION FROM $\Lambda(\mathbb{C}^5)$ AND THE RESULTS OF PAPER I

This paper applies the mirror duality of Ref. [13] (Paper I) to the four-dimensional mirror-matter models SMM4 and SMM4b of Ref. [10]. We first recall what we take from Paper I, then write out the groups with three unitary blocks on which the alternative readings of Sec. VII rest, and finally describe one generation as the exterior algebra of the split tangent space $\mathbb{C}^5=\mathbb{C}^2\oplus\mathbb{C}^3$.

### A. What is taken from Paper I

The Standard Model gauge group $G_{\rm SM}=S(U(2)\times U(3))\cong(SU(3)\times SU(2)\times U(1))/\mathbb{Z}_6$ is Langlands self-dual: its root datum is isomorphic to its dual, and the isomorphisms form a torsor under $\mathrm{Aut}=W\times\{\pm1\}$, with exactly 24 elements (Sec. II of Paper I). Each isomorphism $\Phi$ defines a per-sector electric–magnetic map $S_\Phi$. On the product $T[G_{\rm SM}]\otimes T[G'_{\rm SM}]$ of an ordinary and a mirror sector, Paper I (Sec. III) defines the mirror elements $\mathcal{M}=\mathrm{swap}\circ(S_\Phi\otimes S_{\Phi'})$ and $\mathcal{M}_P=\mathrm{swap}\circ P\circ(S_\Phi\otimes S_{\Phi'})$, together with $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_P$, in which the twist classes of $\Phi$ and $\Phi'$ differ. At $\theta=0$ their fixed loci are $g_3g'_3=g_2g'_2=4\pi$ and $g_Yg'_Y=12\pi$. In the S-dual frames $D$ and $D'$ of the mirror sector, every such element acts classically, and an element is compatible with one chiral generation per sector if and only if it is of the $\mathcal{M}_P$ type or of the $\tilde{\mathcal{M}}$ type: in frame $D$, $\mathcal{M}_P$ becomes the mirror parity $X_{\rm cl}=\mathrm{swap}\circ(P\otimes P)$, and in frame $D'$, $\tilde{\mathcal{M}}$ becomes $\mathrm{swap}\circ(C\otimes C)$. Line operators and their 1-form symmetries are treated in Sec. IV of Paper I, couplings and phenomenological options in Sec. V. Throughout, as in Paper I, the strong–weak element $S$ is a formal map on couplings, charge lattices and line operators; S-duality is expected to be exact only in the $\mathcal{N}=4$ theory, and the supersymmetric settings suggested by the mirror models, including the obstructions to four-dimensional supersymmetry with gauge-charged supercharges, are collected in App. D. No portal operators between the sectors are included.

### B. Groups with three unitary blocks

The readings of SMM4 with gauge group $G_{\rm SM}\times U(2)$ per sector (Sec. VII) and the flavor group of SMM4b rest on the groups $S(U(2)\times U(3)\times U(k))$. For every $k\geq1$ the map

$$\Phi_k(A,B,C)=\big((A/\det C,\ B\det C),\ C\big)\qquad(1)$$

is an isomorphism of Lie groups $S(U(2)\times U(3)\times U(k))\to S(U(2)\times U(3))\times U(k)$; it needs only $2\cdot(-1)+3\cdot1=1$. Hence

$$\begin{aligned}S(U(2)\times U(3)\times U(2))&\cong G_{\rm SM}\times U(2),\\ S(U(2)\times U(3)\times U(6))&\cong G_{\rm SM}\times U(6),\end{aligned}\qquad(2)$$

with $U(2)=(U_{BL}\times SU_I(2))/\mathbb{Z}_2$ and $U(6)=(U_{BL}\times SU_f(6))/\mathbb{Z}_6$. The isomorphisms mix the abelian factors and are not unique, so the physical identification of the factors is fixed by the matter, not by the group. Both groups in Eq. (2) are Langlands self-dual: an explicit root-datum isomorphism is obtained by transporting the self-duality of $G_{\rm SM}$ along $\Phi_k$ and taking the identity on $U(k)$; it maps roots to coroots and is not unique.

### C. One generation from the split tangent space

Following Refs. [11, 12], one generation is the exterior algebra $\Lambda(\mathbb{C}^5)$ of $\mathbb{C}^5=\mathbb{C}^2\oplus\mathbb{C}^3$, on which $G_{\rm SM}=S(U(2)\times U(3))\subset SU(5)$ acts. With the hypercharge embedding $y=6Y=3$ on $\mathbb{C}^2$ and $-2$ on $\mathbb{C}^3$, the even part $\Lambda^{\rm even}$ consists of the 15 left-handed Weyl fields of one Standard Model generation plus $\nu^c$, and $\Lambda^{\rm odd}$ of their CPT conjugates; $\nu^c=\Lambda^0$ is the Fock vacuum and $\nu_R=\Lambda^5$. Standard $B-L$ is $1-\frac{2}{3}N_3$ on all 32 states, where $N_2$ and $N_3$ count the occupied directions in $\mathbb{C}^2$ and $\mathbb{C}^3$. We take the $U_{BL}$ of Ref. [10] to act like $SU_L(2)$ on the left-handed doublets $Q$ and $L$ only (charges $\frac{1}{3}$ and $-1$) and not on $u^c,d^c,e^c,\nu^c$; then

$$U_{BL}=\Big(1-\tfrac{2}{3}N_3\Big)\,\tfrac{1}{2}\big(1-(-1)^{N_2}\big)=(B-L)\,P_{N_2=1},\qquad(3)$$

that is, standard $B-L$ on the $SU_L(2)$-doublet states and zero elsewhere. It commutes with $G_{\rm SM}$ but is not in the Lie algebra $\mathfrak{so}(10)\oplus\mathbb{C}1$ acting on $\Lambda(\mathbb{C}^5)$, and $e^{3\pi iU_{BL}}=(-1)^{N_2}$ is the action of $-1_2\in SU_L(2)$. When $SU_L(2)$ acts on $Q$ and $L$, it is the action induced by $\mathbb{C}^2$; when $SU_L(2)$ acts on $L$ and $SU_I(2)$ on $Q$ (the reading R2 of Sec. III A), neither $SU(2)$ is induced by a linear action on $\mathbb{C}^5$. With 16 gauge fields per sector, one generation gives $n_b=n_f=32$ on shell. With the shared neutrinos $\nu_L=-\nu'_L$, $\nu_R=\nu'_R$ of Ref. [10] the two sectors have $16+16-2=30$ distinct Weyl fields, $n_f^{\rm tot}=60$ and $n_b^{\rm tot}=64$. These boson numbers count 16 gauge fields; in the one-SU(2) reading of Sec. III there are 12 per sector. Table I collects the counts of degrees of freedom used in this paper for the different readings. We count on shell: two per massless vector, three per massive vector, two per Weyl fermion and one per real scalar.

*The gauge supermultiplet of SMM4.* Reference [10] describes the gauge fields and one generation of SMM4 as a

TABLE I. Per-sector counts of degrees of freedom of the readings used in this paper, on shell. "Gauge" is the number of gauge fields, "Weyl" the number of left-handed Weyl fields the sector owns (per generation; for SMM4b, for three generations), with $\nu^c$ included unless stated. The parentheses have two meanings. For the one-SU(2) reading, $n_f = 30$ counts each shared neutrino once, in the sector in which it is a doublet component (Sec. III B). SMM4b is given per sector for three generations, as a degree-of-freedom match and, in parentheses, as a count of distinct states with all 72 modes of $\mathcal{Q}$ and the singlet neutrinos included (Sec. IV B), not the count with the eaten modes counted once of Table III.

| Reading | Gauge | Weyl | $n_b/n_f$ |
|---|---|---|---|
| SMM4 of Ref. [10], no $\nu^c$ | 15 | 15 | 30/30 |
| $\Lambda(\mathbb{C}^5)$ with $SU_I(2)$ and $U_{BL}$ | 16 | 16 | 32/32 |
| One SU(2), $G_{\rm SM}$ | 12 | 16 | 24/32 (24/30) |
| Bidoublet with $U_{BL}$ | 16 | 22 | 32/44 |
| Link field, scalar $V$ (8 real) | 16 | 16 | 40/32 |
| SMM4b, $n_g = 3$ | 12 | 48 | 90/90 (96/96) |

gauge supermultiplet. Neither realization that we tested works. Spacetime supersymmetry would need a supercharge that maps the adjoint, which contains a color octet, onto the conjugate generation, which does not, and no $U(1)$ charge achieves this. An internal Lie superalgebra with the generation and its conjugate as odd part admits no nonzero odd bracket, and only central ones once a central element is added; such superalgebras are not spacetime symmetries of an S matrix [18–20]. So $n_b = n_f$ holds only as a count; the four-dimensional obstructions to gauge-charged supercharges are given in App. D 2.

## III. SMM4 IN THE ONE-SU(2) READING

### A. The one-SU(2) reading of SMM4

We let $SU_I(2)$ act on the left-handed quarks and $SU_L(2)$ on the left-handed leptons, and write representations as $(\mathbf{c}, \mathbf{d}_L, \mathbf{d}_I)_Y$ under $SU_c(3) \times SU_L(2) \times SU_I(2)$, with hypercharge normalized by $Q = T_3 + Y$; the mirror sector is the parity image, with $SU'_I(2)$ and $SU'_R(2)$ on the right-handed mirror quarks and leptons. If $SU_L(2)$ and $SU_I(2)$ are two independent gauge groups (we call this reading R2), each sector is anomalous: the mixed anomaly coefficients are $A_{SU_L^2 Y} = -\frac{1}{4}$ and $A_{SU_I^2 Y} = +\frac{1}{4}$, and both $SU(2)$ factors have a global anomaly [21], with one and three Weyl doublets, respectively.

In the reading we adopt as the main SMM4 scenario, $SU_L(2)$ and $SU_I(2)$ are the lepton half and the quark half of one gauge $SU(2)$: a single gauge field, $A^L_\mu = A^I_\mu$ for connection-normalized fields, under which leptons and quarks are doublets. With $\tau = \theta/2\pi + 4\pi i/g^2$ for each factor, the coupling and angle of this $SU(2)$ are

$$\frac{1}{g_2^2} = \frac{1}{g_L^2} + \frac{1}{g_I^2}, \qquad \theta_2 = \theta_L + \theta_I, \qquad \tau_2 = \tau_L + \tau_I. \quad (4)$$

Since there is a single gauge field, the two kinetic terms add, and only $g_2$ and $\theta_2$ are physical; $g_L$ and $g_I$ only record how the coupling is split between the lepton and quark halves and are not separately observable. One generation then becomes exactly one Standard Model generation, the scalars $\phi_u, \phi_d$ of SMM4 become doublets $(\mathbf{1}, \mathbf{2})_{\pm 1/2}$ of $SU_c(3) \times SU(2)$, and the lepton Yukawa coupling $Le^c\phi_d$ is allowed at dimension four. The theory is anomaly free with no added matter: $A_{SU(2)^2 Y} = -\frac{1}{4} + \frac{1}{4} = 0$, with four Weyl doublets. A rotation of the leptons alone, or of the quarks alone, is not a symmetry, not even a global one, because the quarks fix the automorphism of the single gauge field and the commutant of $\mathfrak{su}(2)$ on a doublet consists of multiples of the identity. The separate R2 anomalies are therefore not anomalies of any symmetry of this theory.

The kernel of the action of $SU_c(3) \times SU(2) \times U(1)$ on this content is $\mathbb{Z}_6$, generated by $(\omega, -1, e^{2\pi i/6})$ with $\omega = e^{2\pi i/3}$ and the $U(1)$ element $e^{i\alpha}$ acting as $e^{6iY\alpha}$. The gauge group of each sector is therefore $G_{\rm SM} = (SU(3) \times SU(2) \times U(1))/\mathbb{Z}_6$, with $X^*/Q \cong \pi_1 \cong \mathbb{Z}$ and exactly 24 root-datum isomorphisms $X^* \to X_*$: it is Langlands self-dual. The identified mirror sector is the parity image of one Standard Model generation, so the two-sector theory is $G_{\rm SM} \times G'_{\rm SM}$ with one generation per sector. All two-sector mirror elements constructed for that setting in Ref. [13] are then available (with $S$ formal), including the $S$-type $\mathcal{M}$ and $\mathcal{M}_P$ with $g_2 g'_2 = 4\pi$ and $g_Y g'_Y = 12\pi$ at the self-dual point; a per-sector $S$ built from any of the 24 isomorphisms preserves the charge lattice $X_* \oplus X^*$ and maps every Wilson line to an 't Hooft line.

Fields added to complete R2, if kept, become vector-like under $G_{\rm SM}$ for the minimal completions; of 36 completions that preserve the $\mathbb{Z}_6$ quotient, 19 become vector-like and 17 stay chiral, all anomaly free under $G_{\rm SM}$. If the minimal completion that respects $SU_I(2)$ acting on color triplets only is kept massless, the kernel shrinks to $\mathbb{Z}_2$ and the group has $X^*/Q = \mathbb{Z} \oplus \mathbb{Z}_3 \neq \pi_1 = \mathbb{Z}$, which is not self-dual.

With three generations per sector, as in SMM4b, the one-SU(2) reading keeps these properties: every triangle anomaly vanishes, the Witten count is 12, and the center kernel has order 6, so that each sector is again $G_{\rm SM}$ with no added matter.

### B. Shared neutrinos and the two-sector content

In Ref. [10] the neutrinos are shared between the sectors, $\nu_L = -\nu'_L$ and $\nu_R = \nu'_R$. We read this as follows: the mirror doublets are right-handed, so $\nu_L$ is a mirror singlet and $\nu_R$ is the neutral component of the mirror doublet $L'_R$; neither neutrino carries charges of both sectors. The sharing is therefore a relabeling: the singlet neutrino of each sector is the doublet neutrino of the other, and the two-sector content is exactly one Standard Model generation per sector, $15 + 15$ Weyl fields,

with no gauge-singlet neutrino. This is why it is anomaly free and self-dual without further work, and why a neutrino Dirac mass is a cross-sector operator, of the type $(LH)(L'H')$, i.e. a portal (cf. Refs. [22, 23], where ordinary and mirror neutrinos are distinct fields mixed by cross-sector mass terms). A Majorana mass of the doublet neutrino of a sector is instead a per-sector $\Delta L = 2$ effect, such as the operator of Ref. [24], and is excluded only if $B - L$ is conserved in each sector (up to its gravitational anomaly; Sec. VI A). With only the neutrinos shared, the two-sector gauge group $G_{\rm SM} \times G'_{\rm SM}$ has kernel $\mathbb{Z}_6 \times \mathbb{Z}'_6$ on the content, of order 36, and is Langlands self-dual. All perturbative anomalies, the mixed $U(1)$ and gravitational ones included, and both Witten anomalies vanish. Of the classical frame forms of the mirror elements, exactly $X_{\rm cl} = {\rm swap} \circ (P \otimes P)$, the frame-$D$ form of $\mathcal{M}_P$, and ${\rm swap} \circ (C \otimes C)$, the frame-$D'$ form of $\tilde{\mathcal{M}}$, map the content to itself.

The literal label map $\psi_L \to -\psi'_L$, $\psi_R \to \psi'_R$ suggested by the neutrino identification is not compatible with the gauge group: for none of the eight choices of the identity or conjugation on each factor does it intertwine $G_{\rm SM}$ with $G'_{\rm SM}$, since it sends the doublet $(\nu_L, e_L)$ to mirror singlets. The gauge-compatible map is $\mathcal{M}_P$ in frame $D$. It exchanges each ordinary multiplet with the conjugate mirror multiplet and acts on the shared fields as $\nu_L \to \eta_L \nu^c$, $\nu^c \to s_\nu \eta_\nu \nu_L$, where $s_\nu = -1$ is the sign in $\nu'_L = s_\nu \nu_L$ and the $\eta$'s are multiplet phases. It descends to the shared content if and only if $\eta'_\nu = s_\nu \eta_L$ and $\eta'_L = s_\nu \eta_\nu$, and then squares to $s_\nu \eta_L \eta_\nu$ on the shared pair. No component of the content is neutral under both $G_{\rm SM}$ and $G'_{\rm SM}$, so every gauge-compatible map is block off-diagonal between the sectors.

Per generation, the two sectors have 30 distinct Weyl fields, i.e. 60 fermionic degrees of freedom, against 64 without sharing. Assigning each shared neutrino to the sector in which it is a doublet component, each sector has an effective $n_f = 30$ per generation, $n_f = 90$ for three generations, while its 12 massless gauge fields give $n_b = 24$. The ultraviolet theory therefore has $n_b - n_f = 24 - 90 = -66$ per sector for three generations; the balance of bosons and fermions of SMM4b (Sec. IV) is a property of the infrared. Counting its own fields only, a sector has $n_b = 24$ and $n_f = 32\, n_g$, with $n_g$ the number of generations.

### C. Quark sharing

We next ask whether the mirror duality requires or allows quarks to be shared as well, so that each sector has 24 effective fermionic degrees of freedom. Besides the neutrinos, 6 quark Weyl fields, two color triplets, must then be shared: two of the four ordinary left-handed triplets $\{u_L, d_L, u^c, d^c\}$ are identified with two of the four mirror ones $\{(u'_R)^c, (d'_R)^c, u'_L, d'_L\}$, which gives 72 patterns, each tested with every color bijection. In $G_{\rm SM} \times G'_{\rm SM}$ no pattern is gauge invariant, since every quark component is charged under both $SU_c(3)$ and $SU'_c(3)$, and none of the 168 single-component sharings is either (sharing $e^c = e'$, a lepton control, is allowed). Identifying the two color groups by complex conjugation admits exactly two patterns, $u^c = u'_L$, $d^c = d'_L$ and $u^c = d'_L$, $d^c = u'_L$ (the identity admits none), at the cost of a colored mirror sector that shares 8 gluons; sharing $Q$ would require $SU(2) = SU(2)'$ as well, and then $\nu_L$ would be a doublet component and a singlet of the diagonal $SU(2)$ at once.

The mirror duality neither requires nor allows quark sharing. It does not require it, since the neutrino-sharing content above is self-dual, anomaly free and preserved by $X_{\rm cl}$ and by the frame form of $\tilde{\mathcal{M}}$. It forbids it in the reading where each sector is $G_{\rm SM}$, and in the color-identified extension the identification fails every test: the diagonal color group has $A_{SU(3)^3_D} = \pm 2$ and there are mixed $U(1)$ anomalies, whose minimal repair gives at least 30 distinct Weyl fields rather than 24; the group admits no self-duality; only ${\rm swap} \circ (P \otimes P)$ and ${\rm swap} \circ (C \otimes C)$ preserve the content; no $S$-type element descends, since ${\rm swap} \circ (S_F \otimes S'_F)$ maps the electric identification lattice to magnetic charges; and an $S$ on the single $\tau_3$ would need $\alpha_3 = 1$, or $\alpha_3 = 2/\sqrt{3}$ at $\theta_3 = \pi$. Quark sharing therefore survives only as a classical, anomalous extension with a non-self-dual gauge group, and it does not reach the count of 24.

### D. The condensate-sharing alternative: shared neutral quark composites

In the condensate-sharing scenario, besides the neutrinos, neutral quark condensates are shared between the sectors, twelve degrees of freedom of them, so that each sector has $n_f = 32 - 2 - 6 = 24$. Since elementary quarks cannot be shared (Sec. III C), the shared objects must be composites, to which the no-go above does not apply. We test this alternative and summarize the results; they are negative or a matter of bookkeeping.

The quark bilinears of one sector are built from $Q$, $u^c$, $d^c$ and their conjugates: Lorentz scalars and $(1,0)$ tensors $\chi_X \chi_Y$, and vectors $\chi_X^\dagger \bar\sigma^\mu \chi_Y$. None of the scalars or tensors is a singlet of $SU(3) \times SU(2)$. The color-singlet scalars are exactly two doublets,

$$\mathcal{H}_u = Q\, u^c \sim (\mathbf{1}, \mathbf{2})_{-1/2}, \qquad \mathcal{H}_d = Q\, d^c \sim (\mathbf{1}, \mathbf{2})_{+1/2}. \tag{5}$$

These composite doublets have the quantum numbers of the conjugate Higgs doublets, $\mathcal{H}_u \sim \phi_u^*$ and $\mathcal{H}_d \sim \phi_d^*$. Their neutral components, the operators $\mathcal{H}_u^0 = \sum_a u_{La} u_a^c$ and $\mathcal{H}_d^0 = \sum_a d_{La} d_a^c$, are invariant only under $SU(3) \times U(1)_{\rm em}$; we write $\langle \mathcal{H}_{u,d}^0 \rangle$ only for their vacuum values. The singlets of $SU(3) \times SU(2)$ are vector currents only, and $J_Q = Q^\dagger \bar\sigma^\mu Q$, $J_u$ and $J_d$ are the only $G_{\rm SM}$ singlets. A count of "twelve" therefore requires a choice of set and of unit. Among the natural sets of bilinears and units, the only $G_{\rm SM}$-invariant set with twelve is the three

TABLE II. The three readings of the shared neutral quark composites in the condensate-sharing scenario: the shared objects, the phase in which their sharing is gauge invariant, and the per-sector counts $n_b/n_f$ for one generation with the shared neutrinos counted once. S2a and S2b count the shared objects by their constituent Weyl fields; $\mathcal{H}^0_{u,d}$ are the neutral components of Eq. (5).

| Reading | Shared objects | Phase | $n_b/n_f$ |
|---|---|---|---|
| S1 | $\mathcal{H}^0_u, \mathcal{H}^0_d$ (3 physical scalars) | broken | $27/\frac{57}{2}$ |
| S2a | $\mathcal{H}^0_u$ or $\mathcal{H}^0_d$ | broken | 27/24 |
| S2b | $J_Q$ or $J_u + J_d$ | unbroken | 24/24 |

singlet currents counted off shell, and no set gives 12 for one generation and 18 for three generations in the same unit: every 12 becomes 108.

A composite $O$ can be shared, $O = \zeta O'$ with $|\zeta| = 1$, only if it is invariant under the gauge group of its sector: in the unbroken phase only $J_Q$, $J_u$ and $J_d$ qualify, and after electroweak breaking also the neutral color singlets $\mathcal{H}^0_u$, $\mathcal{H}^0_d$. We count a shared set as the shared neutrinos are counted: its $X$ degrees of freedom are counted once for the two sectors together, i.e. half in each, and since the composites are made of the sector's quarks, they are subtracted from its fermionic count. Sharing a set of $X$ counted degrees of freedom thus gives $n_f = 30 - X/2$ per sector, with the shared neutrinos counted once, and $n_b = 24$ in the unbroken phase or $n_b = 27$ after the breaking. We carry the three readings of Table II: the neutral scalars counted as such (S1), and the shared objects counted by their constituent Weyl fields, either the scalars after the breaking (S2a) or the currents in the unbroken phase (S2b). The rules that count a shared composite by its own degrees of freedom or by its constituents are postulates of the scenario, not derived.

In S1 one of the four neutral real scalars is the Goldstone mode eaten by the $Z$, already included in $n_b = 27$; the count with the three physical scalars is the one consistent with $n_b = 27$, and its $n_f$ is not an integer. S2b removes its constituents from the elementary content, and the removal is $G_{\rm SM}$-anomalous: removing constituents is never both $G_{\rm SM}$-covariant and anomaly free. Sharing bosonic composites, as in S1, leaves all anomalies, ordinary, mirror and mixed, at zero. Thus $n_b = n_f = 24$ together with gauge-invariant sharing holds only for S2b and for the three $G_{\rm SM}$-singlet currents counted off shell; the latter sets off-shell bosonic components against on-shell fermionic ones, and the shared objects are currents, whose expectation values would break Lorentz invariance. On a shared composite the identification $O = \zeta O'$ is compatible with $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ for suitable quark multiplet phases, but no $S$-type element respects it on $\mathcal{H}^0$, which carries a nonzero weight on the torus of $G_{\rm SM}$, for the same reason as for shared quarks (Sec. III C). None of the readings carries an $\mathcal{N} = 1$ supersymmetry: S1 and S2a have $\mathrm{Str}\,1 = -\frac{3}{2}$ and 3 per sector, and S2b, which balances, 24|24, has $\mathrm{Str}(\omega 1_3) = 27$ and would need the Standard Model fermions to be gauginos; in superfield language the shared objects are not lowest components.

The four-fermion dynamics of Sec. V decides which of these objects can form and be shared. The scalars of S1 and S2a arise only in the left–right channels of the operators $O_{qu}$ and $O_{qd}$, while the currents of S2b appear only in vector channels and never condense, $\langle J^\mu \rangle = 0$. Identifying the composite fields of the two sectors, $\mathcal{H}' = \zeta\mathcal{H}$ with $|\zeta| = 1$, in the auxiliary-field form of the four-fermion interactions gives

$$\mathcal{L}_{\rm eff} = \frac{GG'}{G+G'} \left| O + \zeta^* O' \right|^2, \tag{6}$$

whose cross term is a cross-sector four-fermion operator, i.e. a portal. No product of one ordinary and one mirror scalar composite is invariant under $G_{\rm SM} \times G'_{\rm SM}$; the lowest invariant ones are eight-fermion operators $(\mathcal{H}^\dagger\mathcal{H})(\mathcal{H}'^\dagger\mathcal{H}')$, a composite Higgs portal, and for S2b $J_Q \cdot J'_Q$ is a dimension-six portal. Without portals the sharing can therefore only be bookkeeping: the vacuum values are identified, and the fields are not. This has a direct consequence for the motivation of the scenario. Sharing a vacuum value keeps every quark mass of that type equal in the two sectors, so the vacuum asymmetry $\varepsilon = (v' - v)/v$ of Sec. VI E vanishes: $\varepsilon = 0$ in S1, where both quark types are shared, and on the shared type in S2a; only S2b leaves $\varepsilon$ free. For S1 the $n$–$n'$ window of Ref. [9] is therefore out of reach.

### E. Six-dimensional lift and the ten-dimensional quark space

In Ref. [10] the scalars $\phi_u, \phi_d$ arise as internal components $A_z$ of a six-dimensional gauge field on $T^2/\mathbb{Z}_2$, in the spirit of gauge–Higgs unification [25]. In the lift of reading R2, with $SU_I(2) \subset SU(3)_I$ and $SU_L(2)$ apart, the identification $A^L = A^I$ is not a consistent constraint on the six-dimensional fields: together with the coset fields of $SU(3)_I$, whose $A_z$ are $\phi_u, \phi_d$, it generates all of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(3)_I$ under commutators. The one-SU(2) reading lifts instead along a unified route U, with gauge algebra that of $SU(3)_c \times SU(3)_W[\times U(1)_A]$, orbifold parity $P = \mathrm{diag}(1, 1, -1)$ and $Y = \hat{X}/6 + A$, where $\hat{X} = \mathrm{diag}(1, 1, -2)$ is the generator of $SU(3)_W$ that commutes with $P$ and $A$ that of $U(1)_A$. The zero modes are $SU(3)_c \times SU(2) \times U(1)_X[\times U(1)_A]$ with $\phi_u = (A^{13}_z, A^{23}_z) \in (\mathbf{1}, \mathbf{2})_{1/2}$ and $\phi_d = (A^{31}_z, A^{32}_z) \in (\mathbf{1}, \mathbf{2})_{-1/2}$, doublets of the one $SU(2)$ of leptons and quarks. Bulk triplets with intrinsic parity $+1$ give $(\mathbf{3}, \mathbf{3})_0 \to Q + d^c$, $(\mathbf{3}, \bar{\mathbf{3}})_{1/3} \to Q + u^c$, $(\mathbf{1}, \mathbf{3})_{-2/3} \to L + e^c$ and $(\mathbf{1}, \bar{\mathbf{3}})_{-1/3} \to L + \nu^c$, and the gauge coupling restricted to the zero modes gives one Yukawa coupling per bulk triplet, of type II when the Yukawa triplets share one six-dimensional chirality. At tree level

$$\frac{1}{g_Y^2} = \frac{1}{3g_2^2} + \frac{1}{g_A^2}, \qquad \hat{\tau}_Y = \tau_2 + 3\tau_A, \tag{7}$$

so that without $U(1)_A$ one has $\hat{\tau}_Y = \tau_2$, i.e. $\sin^2\theta_W = \frac{3}{4}$ at the compactification scale, the known value of $SU(3)_W$ gauge–Higgs unification [26]; reaching the observed value needs a further contribution to the hypercharge coupling, such as the $U(1)_A$ term of Eq. (7), which lowers $g_Y$ at fixed $g_2$. The six-dimensional group is $G_6^U = (SU(3)_c \times SU(3)_W \times U(1)_A)/\mathbb{Z}_3$ (or without the $U(1)_A$ factor): the $\mathbb{Z}_6$ of the Standard Model maps onto this $\mathbb{Z}_3$ with kernel $\mathbb{Z}_2$, and on one generation with $\phi_u, \phi_d$ the four-dimensional kernel has order 6, so each sector is $G_{\rm SM}$.

The anomaly-free completions of route U, with six- and four-dimensional anomalies and Green–Schwarz terms [27, 28], confirm that one Standard Model generation with all fields on branes needs no added matter, so the statement of Sec. III A survives the lift; obtaining all three Yukawa couplings $Qd^c$, $Qu^c$, $Le^c$ from $A_z$ costs 16 added Weyl components without supersymmetry, and every optimum keeps the kernel $\mathbb{Z}_6$, so each sector is $G_{\rm SM}$ at every Kaluza–Klein level.

The lift requires $\mathrm{Im}\,\hat{\tau}_Y \geq \mathrm{Im}\,\tau_2$, i.e. $g_Y^2 \leq 3g_2^2$. On the $\theta = 0$ slice the per-sector $S$ of the $S$-type elements ($g_2 \to 4\pi/g_2$, $g_Y \to 12\pi/g_Y$) maps $g_Y^2 < 3g_2^2$ to $g_Y^2 > 3g_2^2$, so these elements relate lifted sectors only on the boundary $g_Y^2 = 3g_2^2$ ($1/g_A^2 = 0$), which contains the point $\alpha_2 = 1$, $\alpha_Y = 3$ fixed by $S$ and $X_{\rm cl}$. Without $U(1)_A$ the lift is mapped to itself by $S$, $T$ and $P$, and the classical elements survive in either variant. In the lift of reading R2 the one-loop potential of a constant Wilson line along the broken direction has local minima only at the origin and at the half and third periods, so the Wilson line cannot produce a vacuum value much below the compactification scale. The lift of the one-SU(2) reading has a potential of the same form; its origin, where $G_{\rm SM}$ is unbroken, is a strict minimum for the brane-only content, and a minimum at half a period breaks $SU(2) \times U(1)_Y$ to $U(1)_{\rm em} \times U(1)'$ at the compactification scale, so a small electroweak scale needs another source. In the ten-dimensional quark space $X_6 = (T^2)^3/\Gamma$ of Ref. [10], read along route U, the unbroken algebra is $\mathfrak{su}(2) \oplus \mathfrak{u}(1)_X$ for every twist; no chiral generation comes from the bulk for $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, while for $\Gamma = \mathbb{Z}_3$ one triplet or antitriplet gives exactly three generations, with a tree-level Yukawa matrix of rank 2, and the ten-dimensional reflection realizes $P \otimes P$, the frame-$D$ form of $\mathcal{M}_P$, and its composition with the conjugation of $SU(3)_W$ realizes $CP \otimes CP$, the frame-$D'$ form.

### F. Strong–weak duality in the one-SU(2) reading

As in Paper I, the per-sector $S$ in $\mathcal{M}$ and $\mathcal{M}_P$ is a formal map; the reasons, given in App. B of Ref. [13], apply unchanged to the one-SU(2) reading, and we only summarize them. Since the $SU(2)^2$ anomaly coefficient of $B+L$ is $n_g \neq 0$, a $B+L$ rotation shifts $\theta_2$, and a factorwise $S$ maps equivalent couplings to inequivalent ones. The self-dual couplings $\alpha_3 = \alpha_2 = 1$ and $\alpha_Y = 3$ are never reached at a common scale: in a formal one-loop extrapolation of the Standard Model, $\alpha_Y = 3$ is reached only at $1.3 \times 10^{41}$ GeV, above the Planck mass, so $\mathcal{M}_P$ must be read in the $S$-dual frame or only at an ultraviolet point. An exact duality is available in a vector-like $\mathcal{N} = 2$ tool, in which every chiral field goes into a hypermultiplet and $SU_c(3)$ and $U(1)_Y$ are weakly gauged spectators: $SU(2)$ with $N_f = 4$, whose $SL(2,\mathbb{Z})$ duality acts together with $Spin(8)$ triality [29]. For one generation the tool is conformal only if $\phi_d$ is removed or identified with $L$; with $SU_c(3)$ and $U(1)_Y$ gauged its exact dualities form $\Gamma^0(2)$, whose strong–weak element $TST^{-1}$ fixes $\theta_2 = \pi$, $g_2^2 = 8\pi$ (with $\tau_{SW} = 2\tau_2$ in the normalization of Ref. [29]), so the $S$ of the tool, $S_{SW}$, is never available and the mirror elements built from $TST^{-1}$ hold at $g_2 g_2' = 8\pi$. None of $\mathcal{M}$, $\mathcal{M}_P$, $\tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}_P$, which act as $\tau_{SW} \to -4/\tau_{SW}$, is realized, the vector-like tool leaves the $B+L$ obstruction untouched, and for $n_g = 3$ it has $b_0(\mathfrak{su}_2) < 0$, so neither $S_{SW}$ nor $TST^{-1}$ carries over to SMM4b.

Paper I states only the result of the same tool on $SU_I(2)$ alone, in the reading R2 without $U_{BL}$ (Sec. III A); in brief, with the hypermultiplets of $Q$ and of $(\phi_u, \phi_d)$, $N_f = 4$ and $b_0(SU_I(2)) = 0$. On the global forms $(SU(3) \times SU_L(2) \times SU_I(2) \times U(1)_Y)/\mathbb{Z}_n$, $S_{SW} : \tau_I \to -1/(4\tau_I)$ is excluded for $n = 2, 6$, and for $n = 1, 3$ it preserves the line lattice but is not a duality of the tool with $\mathfrak{su}(3)_c \oplus \mathfrak{u}(1)_Y$ gauged, since it maps the quark class $(0,1)$ to the monopole class $(1,0)$, which carries the spinor of $Spin(8)$ that does not restrict to the quark representations (the bad spinor). The dualities that preserve the gauged spectators form $\Gamma^0(2)$, whose strong–weak element $TST^{-1}$ fixes $\theta_I = \pi$, $g_I^2 = 8\pi$, with $g_I g_I' = 8\pi$; with three generations there is no conformal point, and neither $S_{SW}$ nor $TST^{-1}$ carries over.

The completion by hypermultiplets is also a model in its own right, with a physical meaning for the mirror sector. In an $\mathcal{N} = 2$ theory with a single $G_{\rm SM}$ and one hypermultiplet for each Standard Model chiral field, the hypermultiplet partner of each Standard Model fermion, written as a field of opposite chirality, is exactly its mirror fermion of SMM4: the two sectors become superpartners under one gauge group. The model is anomaly free for one and three generations and admits the global form $G_{\rm SM}$. For one generation without Higgs hypermultiplets its $SU(2)$ is the conformal tool above, with the duality group $\Gamma^0(2)$ and the strong–weak element $TST^{-1}$. Its Lagrangian has a parity that exchanges each Standard Model fermion with its partner; this is what remains of $\mathcal{M}_P$ on a single $G_{\rm SM}$, without the $S$ part. Unfolding the model into two sectors, a quiver $G_{\rm SM} \times G_{\rm SM}'$ whose bifundamental link Higgses it back to the diagonal, is possible, but no such quiver is symmetric under the exchange of the sectors when the number of generations is odd. With an asymmetric quiver as the model, $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ are dualities between quivers that share the same diagonal theory, rather than symmetries of one quiver. The model, its curve, its parity and the quiver are described

in App. E.

## IV. SMM4B: CONTENT, COUNTING AND PSEUDO-SUPERSYMMETRY

SMM4b is the three-generation model of Ref. [10]. Each sector has the gauge group $G_{\rm SM}$ and three generations of quarks and leptons, and no elementary Higgs field: the six quark condensates play the role of the Higgs fields, one per quark flavor, grouped into three pairs $(\phi_u, \phi_d)$ of doublets. Electroweak symmetry breaking by quark condensates without elementary Higgs fields has been studied in QCD-like worlds, where QCD alone breaks it at the hadronic scale [30]; in SMM4b the condensates are produced at the electroweak scale by the four-fermion interactions of Sec. V. The balance of bosonic and fermionic degrees of freedom that Ref. [10] calls pseudo-supersymmetry is a property of this infrared model; the ultraviolet one-SU(2) model of Sec. III does not have it (Sec. III B). We describe the content and the counting in the neutrino-sharing scenario, in which only the neutrinos are shared, as in Sec. III B, and summarize the condensate-sharing alternative of Sec. III D in Sec. IV C; the dynamics that produces the condensates is the subject of Sec. V, and the action of the mirror maps that of Sec. VI.

### A. Flavor group and condensate

The quark flavor group of SMM4b is the $U(6)$ factor of Eq. (2) with $k = 6$: by $S(U(2) \times U(3) \times U(6)) \cong G_{\rm SM} \times U(6)$, the six quark flavors can carry a $U_f(6) = (U(1) \times SU(6))/\mathbb{Z}_6$ next to the gauge group. In SMM4b this $U_f(6)$ is a global symmetry and is broken; only $G_{\rm SM}$ is gauged. The order parameter is the color-singlet condensate $\mathcal{Q}_{ij} \sim q_{Li}\bar{q}_{Rj}$ of the six quark flavors, on which the chiral flavor group $G_{72} = U_f(6)_L \times U_f(6)_R$, of dimension 72, acts as $\mathcal{Q} \to L\mathcal{Q}R^\dagger$. Under $G_{\rm SM}$ the condensate is

$$\mathcal{Q} = 9\times(\mathbf{1},\mathbf{2})_{+1/2}\,[Q_i\bar{d}_{Rj}] \,\oplus\, 9\times(\mathbf{1},\mathbf{2})_{-1/2}\,[Q_i\bar{u}_{Rj}], \quad (8)$$

18 complex color-singlet doublets, i.e. 72 real fields, on which the $\mathbb{Z}_6$ of $G_{\rm SM}$ acts trivially. The six vacuum values sit in the neutral components of $Q_g\bar{u}_{Rg}$ and $Q_g\bar{d}_{Rg}$, $g = 1, 2, 3$: three pairs $(\phi_u, \phi_d)$. These doublets break $SU(2) \times U(1)_Y \to U(1)_{\rm em}$, and three of their Goldstone modes, two charged and one neutral, are eaten by $W^\pm$ and $Z$.

Reference [10] counts 63 pseudo-Goldstone bosons (pNGBs). The Goldstone theorem reproduces this count for the vacuum $\langle\mathcal{Q}\rangle = \mathrm{diag}(w, w, w_c, w_s, w_t, w_b)$ in the flavor order $(u, d, c, s, t, b)$, with $w_c, w_s, w_t, w_b$ distinct and different from $w$; here the $w$'s are entries of the condensate, not the electroweak vacuum value $v$. The unbroken group is $H = U_V(2) \times U_V(1)^4$, of dimension 8, so 64 generators are broken, 28 vector and 36 axial. Exactly one direction of $\mathfrak{u}(6) \oplus \mathfrak{u}(6)$ has an $SU_c(3)^2$ anomaly, the singlet $U_A(1)$, and it gives no Goldstone boson. Hence

$$N_{\rm pNGB} = \dim G_{72} - \dim H - 1 = 72 - 8 - 1 = 63, \quad (9)$$

35 pseudoscalars and 28 scalars. Other degeneracy patterns give other counts: 35 when all six values are equal, 65 when all are distinct, and 61 for $w_u = w_d$, $w_c = w_s$. The count assumes that the vector part $U(6)_V \to U(2) \times U(1)^4$ is broken spontaneously. In vector-like QCD at $\theta = 0$ the Vafa–Witten theorem forbids this [31]; we assume that the chiral weak interactions and the four-fermion interactions of Sec. V evade it. The chains we construct in Sec. V B do not realize this pattern as stated: they break the vector flavor symmetry explicitly, through an $S_3$-invariant input, and in their vacua most of the 63 broken generators are heavy. The count is therefore one of broken generators, i.e. of degrees of freedom, not of light states. Of the 72 real modes of $\mathcal{Q}$, 63 are the pNGBs, one is the $\eta'$ and eight are radial (non-Goldstone) scalars. All pNGBs are color singlets, 29 neutral and 34 charged; after the three eaten modes are removed, 60 physical pNGBs remain, 28 neutral and 32 charged. With $G_{\rm SM}$ gauged, the exact flavor symmetry is only the commutant $U(3)_Q \times U(3)_u \times U(3)_d$ of the gauge generators, and 22 of the 60 would be exact Goldstone bosons; the others are pseudo-Goldstone bosons only through the explicit breaking of $G_{72}$ by the electroweak gauging. The four-fermion couplings of chain N2 (Sec. V) break this $U(3)^3$ explicitly as well: their scalar channels have a symmetry of dimension 6 only.

### B. The neutrino-sharing scenario

Counting degrees of freedom as in Ref. [10] (two per massless vector, three per massive vector, two per Weyl fermion, one per real scalar), the gauge bosons after electroweak breaking give $2 \times 9 + 3 \times 3 = 27 = 24 + 3$, and adding the 63 pNGBs gives $n_b = 90$. The three generations of quarks and leptons without the neutrino singlets give $n_f = 3 \times 2 \times (6 + 3 + 3 + 2 + 1) = 90$. Thus $n_b = n_f = 90$ per sector, as stated in Ref. [10]. This is a match of degrees of freedom, not of distinct states: since the condensates break the electroweak symmetry, the three longitudinal modes are three of the 63 pNGBs, and the number of distinct bosonic degrees of freedom is $24 + 63 = 87$. A count of distinct states is obtained by including all 72 modes of $\mathcal{Q}$ and the singlet neutrinos: $n_b = 2 \times 12 + 72 = 96 = n_f = 2 \times 48$, before and after the breaking. In the split of Ref. [10], one gauge pseudo-multiplet holds 90 bosonic and 90 fermionic states and three chiral pseudo-multiplets $(\nu_R^g; \phi_u^g, \phi_d^g)$ hold two of each; the 90 then counts distinct states if its three extra bosonic modes are taken from the non-Goldstone modes of $\mathcal{Q}$. Which modes these are is a bookkeeping choice. In the choice that keeps one Higgs field per quark flavor, the six Higgs fields are the neutral scalars $\mathrm{Re}\,\mathcal{Q}_{ff}$, one

per quark condensate, and the three extra modes are the $\eta'$ and two modes of $\mathcal{Q}_{ud}$.

The helicity content of one sector, with the condensate counted as 72 real scalars and $\nu_R$ included, is that of $\mathcal{N}=4$ super-Yang–Mills theory with a gauge group of dimension $12 = \dim G_{\rm SM}$: 12 states at each of $h=\pm1$, 48 at $h=\pm\frac{1}{2}$ and 72 at $h=0$, i.e. $(1,4,6,4,1)\times12$. The representations do not match. Supercharges that commute with the gauge group keep a supermultiplet in one irreducible representation, so $\mathcal{N}=4$ needs adjoint fermions and scalars, which SMM4b does not have; accordingly the graded characters do not vanish, $\mathrm{Str}(\omega 1_3)=108$ and $\mathrm{Str}(-1_2)=-48$. The reading "$\mathcal{N}=4$ = one gauge plus three chiral multiplets" has the multiplet types of the $\mathcal{N}=1$ decomposition of $\mathcal{N}=4$ but not their sizes. A fit with the right sizes exists if the **3** of $SU(3)\subset SU(4)_R$ is read as the generation index, but it pairs quarks with condensates through color-triplet supercharges, whose gauging is obstructed in four dimensions (App. D 2). The $\mathcal{N}=4$ of SMM4b is therefore a structure of the state count, and no linear superalgebra acts on it.

Sharing the neutrinos between the sectors breaks the balance further. The two sectors have $n_b=192$ and $n_f=180$ distinct states, so $\mathrm{Str}\,1=+12$: the two gauge pseudo-multiplets still match, 90+90, but the three chiral pseudo-multiplets of each sector lose their fermions, and the 12 Higgs states are unpaired. No component of the content is neutral under both $G_{\rm SM}$ and $G'_{\rm SM}$, so every gauge-compatible mirror map is off-diagonal between the sectors.

### C. The condensate-sharing alternative

The counts of this scenario are introduced in this paper; Ref. [10] gives only $n_b=n_f=90$ for SMM4b. In the condensate-sharing scenario we postulate $n_f = 96-6-18=72$ per sector, with 18 shared degrees of freedom made of six Higgs fields, three modes eaten by $W^\pm, Z$ and nine pNGBs, and $n_b = 24+3+9+36=72$, with 36 pNGBs of a $U_f(6)$ acting on the left-handed quarks. Scaled to three generations, the readings of Table II give, per sector, $n_f = 90 - X/2$ for $X$ shared degrees of freedom: $n_f=72$ in S2a and S2b, and in S1 when all 36 neutral scalars are shared, but $n_f=\frac{145}{2}$ for S1 with the 35 physical ones. The "18" works only as the reduction $X/2=18$; its split into six Higgs fields, three eaten modes and nine pNGBs fails in every reading. On the boson side, the 45 pNGBs are those of $G_{54}=U_f(6)_L\times U(3)_u\times U(3)_d$, the $U_f(6)$ of the left-handed quarks together with the right-handed group that hypercharge allows (the limit $g_2\to0$ with $g_Y$ kept): at the vacuum class of chain N2 with 63 pNGBs of $G_{72}$ (one degenerate pair of same-type quarks), 46 generators of $G_{54}$ are broken, and removing the $\eta'$ leaves

$$N_{\rm pNGB}(G_{54})=45, \qquad n_b=27+45=72, \tag{10}$$

again a match of degrees of freedom, with 69 distinct states. The difference $63-45=18$ is the number of generators that mix $u_R$ with $d_R$, which hypercharge breaks explicitly. Removing the constituents of S2a or S2b literally leaves 14, 12 or no pNGBs, so these readings enter only as bookkeeping. As in the neutrino-sharing scenario the balance is not a supersymmetry ($\mathrm{Str}(\omega1_3)\neq0$), and the Higgs fields that the pseudo-multiplets $(\nu_R^g;\phi_u^g,\phi_d^g)$ would share are massive radial modes of $\mathcal{Q}$, while $\nu_R$ is massless. Table III compares the two scenarios.

TABLE III. SMM4b per sector in the two scenarios (three generations). $n_b$ is the degree-of-freedom match of Ref. [10]; in parentheses, the distinct bosonic states of this count, with the eaten modes counted once (not the count with all 72 modes of $\mathcal{Q}$ of Table I); $n_f$ counts each shared field once. For the condensate-sharing scenario the readings are those of Table II; S1* is S1 with the 35 physical neutral scalars. The pNGB counts are at the vacuum class of chain N2 with one degenerate same-type pair (Sec. V B).

| | neutrino-sharing | condensate-sharing |
|---|---|---|
| Shared | $\nu_L,\nu_R$ | $\nu_L,\nu_R$, composites |
| Flavor group | $G_{72}$ | $G_{54}$ |
| pNGBs (eaten) | 63 (3) | 45 (3) |
| $n_b$ | 90 (87) | 72 (69) |
| $n_f$ | 90 | 72 (S1*: $\frac{145}{2}$) |
| Shared Higgs fields | 0 | 6 (S1), 3 (S2a), 0 (S2b) |

## V. THE FLOW FROM SMM4 TO SMM4B

We now connect the two models dynamically. The ultraviolet theory is the one-SU(2) SMM4 of Sec. III with three degenerate generations per sector, but without the scalars $\phi_u,\phi_d$ of Ref. [10]: gauge fields and chiral fermions only. (SMM4 with these scalars is the two-doublet model of Sec. VI E; the chains and that model are the two cases we study.) The infrared theory is SMM4b of Sec. IV, in which quark condensates break the flavor and electroweak symmetries. In between we add four-fermion interactions within each sector; their attractive channels make the quark bilinears condense, as in the Nambu–Jona-Lasinio (NJL) model [16, 17] and in top condensation [32–34]. Cross-sector four-fermion operators are portals and are excluded. The degeneracy of the generations is lifted by an explicit $S_3$-invariant input at the cutoff $\Lambda$ (chain N2 below). The condensate-sharing alternative shares this per-sector dynamics; it differs in what is shared and in the flavor group used to count the light modes (Sec. V D). All dynamical statements below are at leading order in $1/N_c$ (mean field), without gauge loops unless stated.

### A. Four-fermion operators and condensation

There are 49 Lorentz- and $G_{\rm SM}$-invariant four-fermion contents built from $Q, u^c, d^c, L, e^c, \nu^c$ and their conjugates. The 20 that contain $\nu^c$ are portals, since $\nu^c$ is a component of a mirror doublet. The other 29 are the standard dimension-six four-fermion classes. The flavor group $U(3)^5$ of the gauge theory allows exactly the 15 current–current classes, which conserve $B$, $L$ and the axial $U_A(1)$ of the quarks; a diagonal $U(3)_F$ adds $O_{ledq}$, $O_{lequ}$ and $O_{quqd}$; $S_3$ alone allows all 29, the $B$-violating ones [24] included. The mirror-symmetric Lagrangian is $\sum_i C_i(O_i + O_i')$, where $O_i'$ is the same class built from mirror fields. $\mathcal{M}_P$ maps $O_i$ to $O_i'$, so the coefficients obey $C_i' = C_i$; $\tilde{\mathcal{M}}$ maps $O_i$ to $O_i'^\dagger$, so $C_i' = C_i^*$.

Only the left–right operators give scalar, chirality-flipping channels. For the quark doublet $\bar{u}_R q_L \in (\mathbf{1},\mathbf{2})_{-1/2}$, after a Fierz rearrangement the operators $C_{qu}^{(1)}O_{qu}^{(1)} + C_{qu}^{(8)}O_{qu}^{(8)}$ contain the channel term $\mathcal{L} \supset G_u\,|\bar{u}_R q_L|^2$ with $G_u = -2C_{qu}^{(1)}/N_c - C_{qu}^{(8)}(1-1/N_c^2)$, and $G_d$ likewise from $O_{qd}$. Once the generations are distinguished, $G_t$ ($t = u, d$) is a positive hermitian form on the generation indices of the channel, and we write $K_t = G_t^{-1}$ for its inverse, the coupling matrix that enters the mean-field potential (App. F 1). In a channel with $n$ color components in the loop and a Euclidean cutoff $\Lambda$, the gap equation for the dynamical mass $m$ is

$$1 = G\,\frac{n}{8\pi^2}\Big[\Lambda^2 - m^2\ln\Big(1+\frac{\Lambda^2}{m^2}\Big)\Big], \qquad G_c = \frac{8\pi^2}{n\Lambda^2}, \tag{11}$$

with a unique nontrivial solution, which is the minimum, for $G > G_c$. The quark channels ($n = N_c$) are critical before the lepton ones ($n = 1$), so the quark bilinears condense first, all generations together for $U(3)^5$ operators, and the leptons stay massless unless $O_{ledq}$ or $O_{lequ}$ is present. The composites are the doublets of Eq. (8). The electroweak scale follows from the dynamical masses through the leading-logarithmic form of the Pagels–Stokar relation [35],

$$v^2 = \sum_f \frac{N_c\, m_f^2}{8\pi^2}\ln\frac{\Lambda^2}{m_f^2}, \tag{12}$$

the sum running over the condensed flavors. For illustration, $v = 246.22$ GeV with $\Lambda = 10^{15}$ GeV needs $m = 164.6$ GeV for one condensate, or 66.2 GeV each for six degenerate ones, and a tuning $1 - G_c/G \sim 10^{-24}$. The value for one condensate is the fermion-bubble result of top condensation; the renormalization-group treatment of Ref. [33] gives a heavier top quark at the same cutoff. The chains of Sec. V B are not of this kind: they are untuned, with all channels supercritical, and $v$ fixes their cutoff at $\Lambda \approx 0.85$–$1.2$ TeV (App. F 1).

### B. Vacua and the lifting of the generations

With $U(3)^5$ operators the generations stay degenerate, $\langle\mathcal{Q}\rangle = \mathrm{diag}(U, D, U, D, U, D)$. With $S_3$-invariant couplings, independent for the up and the down channels (hypercharge forbids a mixed term), generic vacua have 61, 63 or 65 pNGBs (in the count sample of App. F 1 and in the other chain-N2 samples there). The count 63 needs no tuning: one quark type splits as $1 + 2$ and the other has three distinct masses, so the degenerate pair lies within one type (for example up masses $(2, 2, 7)$ and down masses $(3, 5, 11)$, with $H \cong U(2) \times U(1)^4$). The isospin-like pairing $w_u = w_d$ of Eq. (9) needs a tuned relation. Holomorphic $O_{quqd}$-type couplings, which $S_3$ allows, split the degenerate pair, so 63 is generic only when they vanish. They are forbidden by an anomaly-free $\mathbb{Z}_3^A$, with charges $(1, 1, 1, 0, 0)$ on $(Q, u^c, d^c, L, e^c)$, which also forbids $O_{ledq}$ and $O_{lequ}$; the condensates carry charge 2, so $\mathbb{Z}_3^A$ is broken spontaneously at condensation. The ultraviolet flavor group is $U(3)_Q \times U(3)_u \times U(3)_d$ at nonzero electroweak couplings; it grows to $U(6)_L \times U(3)_u \times U(3)_d = G_{54}$ for $g_2 \to 0$ and to $G_{72}$ only for $g_2 = g_Y = 0$. The full $G_{72}$ is a symmetry of the quark four-fermion sector only on a subspace defined by eight conditions, among them $G_u = G_d$. Heavy color-octet (Kaluza–Klein gluon) exchange gives such a $G_{72}$-invariant interaction, while the exchange of a heavy vector coupled to hypercharge gives $G_u - G_d > 0$, so that up-type quarks condense first.

Table IV summarizes the candidate chains. Chain N2, with $S_3 \times \mathbb{Z}_3^A$-invariant four-fermion couplings at $\Lambda$, is the one we use: the anomaly-free $\mathbb{Z}_3^A$ protects the absence of $O_{quqd}$, the vacua have 61, 63 or 65 pNGBs, and the generations are lifted by the explicit $S_3$ input. No chain gives the 63 light pNGBs of Eq. (9) with an isospin $SU_V(2)$ and spontaneous vector breaking: in a 63 vacuum of chain N2 the $U(2)$ acts on a same-type pair, not as isospin. The 63 count is a count of broken generators, not of light states: in the $S_3$ vacua of the curvature sample exactly 4 directions are zero modes, and in its 63 vacua the 28 broken vector directions are heavy, with curvature at least of order 0.1 of the heaviest radial mode (0.14 in this sample, a bound that depends on the sampling); only axial directions can be light. The gauge group stays Langlands self-dual at every stage of the aligned branch: $G_{\rm SM}$ with 24 root-datum isomorphisms in the ultraviolet, $U(3)_{c,\mathrm{em}}$, the unbroken color and electromagnetic group, with 12 in the infrared. It fails only on the charge-breaking branch, where the infrared group is $SU(3)_c$, and at the six-dimensional bulk stage of the Kaluza–Klein gluon chain with $U(1)_A$. Electroweak gauge loops favor the $U(1)_{\rm em}$-preserving alignment of the condensates at fixed norms, but in the 23 coupling sets of the gauge-loop sample whose lowest gaugeless minimum is misaligned they overturned none; where that minimum is misaligned, the infrared stays on the charge-breaking branch.

The scalar-channel dynamics has the symmetry $G_{54}$

TABLE IV. Candidate chains from SMM4 to SMM4b: the input at the cutoff $\Lambda$ and whether the degeneracy of the generations is lifted. The mirror elements $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ survive every stage of chains N0–N2 as their frame forms. Chain N2 is the one used in this paper; the near-$G_{54}$ chain of Eq. (13) is a variant of it.

| Chain | Input at $\Lambda$ | Generations |
|---|---|---|
| N0 | $U(3)^5$ NJL | degenerate |
| N1 | KK gluons (route U) | degenerate |
| N2 | $S_3 \times \mathbb{Z}_3^A$ NJL | lifted by explicit $S_3$ |
| N3 | approx. $G_{72}$, near-critical | split explicitly |
| N4 | $U(3)^5$ 8-fermion term | lifted spontaneously |

of Sec. IV C exactly for $U(3)^5$ operators with $G_u \neq G_d$ ($G_{72}$ for $G_u = G_d$), but only a six-dimensional symmetry for the $S_3$-only couplings of chain N2. In the 63 class of chain N2, where 45 generators of $G_{54}$ are broken, these 45 modes are not light: in the main sample of App. F 1, besides 4 zero modes the median curvature is 0.37 of the heaviest radial mode. They become light in a variant, the near-$G_{54}$ chain, in which

$$G_t = g_t\, G_c\, (1 + \epsilon_S D_t), \qquad t = u, d, \tag{13}$$

with $g_u \neq g_d$, a traceless $S_3$-invariant $D_t$ of unit spectral norm, and a small splitting $\epsilon_S$. There the 45 split as 4 zero modes, 32 light modes (curvature below $5\epsilon_S$) and 10 heavy ones (above 0.5), and the generations are split by an amount of order $\epsilon_S$ (the median over quark types of the spread $(\max - \min)/\max$ of the three masses is 0.007, 0.04 and 0.1 at $\epsilon_S = 0.01$, 0.05 and 0.2). This chain needs the tuning $\epsilon_S \ll 1$ beyond that of chain N2. In both chains most 45-class vacua are local minima of the full 72-dimensional potential (40 of 55 in chain N2, 28 of 35 near $G_{54}$, in the main sample); the others are unstable toward $U(1)_{\rm em}$-breaking directions.

*Status of the chains as models.* The chains are built to realize the symmetry-breaking pattern of SMM4b dynamically, and they do not reproduce the observed spectrum.

(i) *Charged leptons.* The only lepton-mass channels are $O_{ledq}$ and $O_{lequ}$, which $\mathbb{Z}_3^A$ forbids, so in chain N2 the electron, muon and tau are massless. A variant fixes this. The $\mathbb{Z}_3$ with charges $(1, 1, 1, 1, 1)$ on $(Q, u^c, d^c, L, e^c)$ is anomaly free, still forbids $O_{quqd}$ and allows exactly $O_{ledq}$, $O_{duql}$ and $O_{qque}$ at dimension six, so the charged leptons get masses through $O_{ledq}\langle Qd^c\rangle$; it acts on the quark condensates and on $O_{quqd}$ as $\mathbb{Z}_3^A$ does, so at leading $N_c$ we take the quark vacua to be those of chain N2. The size of the lepton masses is not computed.

(ii) *Neutrinos.* With the sharing of Sec. III B, a Dirac mass $\nu_L \nu^c$ would be a portal. A Majorana mass of the doublet neutrino of a sector needs a per-sector $\Delta L = 2$ operator, such as the dimension-five operator of Ref. [24] with elementary doublets or $(LQd^c)^2$ at dimension nine without them; it is absent at the dimension six of chain N2 and violates the $B - L$ of each sector, which chain N2 conserves exactly up to its gravitational anomaly (Sec. VI A). In chain N2 the neutrinos are therefore massless.

(iii) *Quark masses.* All six quarks condense with masses of the same order, with no hierarchy. In the main sample of chain N2 (App. F 1), 112 of 120 vacua have all six quarks at 50 GeV or above, and the other 8 have a massless quark; in the near-$G_{54}$ chain all 90 have all six above 246 GeV.

(iv) *Pseudo-Goldstone bosons and electroweak precision.* Only 4 directions are exact zero modes, the vector directions are heavy, and the masses of the 32 charged pseudo-Goldstone bosons are not computed. Nor are the oblique parameters $S$ and $T$ [36, 37]; with up and down condensates split by factors of order one and with these charged states, they need not be small.

(v) *Baryon number.* $\mathbb{Z}_3^A$ allows two of the four $B$-violating dimension-six classes, $QQQL$ and $d^c u^c u^c e^c$, and both mediate $p \to e^+\pi^0$. In the normalization

$$\mathcal{L} \supset C_{LL}\, \epsilon^{abc} (u_a^T \mathsf{C} P_L d_b)(u_c^T \mathsf{C} P_L e) + C_{RR}\, (L \to R) + \text{h.c.}, \tag{14}$$

with $\mathsf{C}$ the charge-conjugation matrix, $a, b, c$ color indices, first-generation fields in the mass basis, $C_{LL} = 2C_{QQQL}$ and $C_{RR}$ fed by $d^c u^c u^c e^c$, the matrix element of Ref. [38] and the bound $\tau(p \to e^+\pi^0) > 2.4 \times 10^{34}$ yr of Ref. [39] give $(|C_{LL}|^2 + |C_{RR}|^2)^{1/2} < 7.4 \times 10^{-32}$ GeV$^{-2}$ at 2 GeV, i.e. $\Lambda_{\rm eff} = C^{-1/2} > (3.69 \pm 0.07_{\rm stat}{}^{+0.21}_{-0.23,\,\rm sys}) \times 10^{15}$ GeV, with both errors from the matrix element; one-loop QCD running from $10^{15}$ GeV raises the bound to $(5.14 \pm 0.10_{\rm stat}{}^{+0.30}_{-0.32,\,\rm sys}) \times 10^{15}$ GeV. Coefficients of the critical size $8\pi^2/(N_c\Lambda^2)$ of the scalar channels would give $\tau \approx 1.9 \times 10^{29}$ yr even at $\Lambda = 10^{15}$ GeV, and on the 45-class vacua of App. F 3, with $\Lambda \approx 0.85$–$1.2$ TeV, the bound requires $C\Lambda^2 \lesssim 10^{-25}$, at most $3.8 \times 10^{-27}$ of the critical size. These order-of-magnitude estimates show that the $B$-violating coefficients must be suppressed far below the condensing ones, by a tuning or by a symmetry beyond $S_3 \times \mathbb{Z}_3^A$; through $C_i' = C_i$ or $C_i' = C_i^*$ the bound applies to the mirror sector as well. The two operators do not enter the leading-$N_c$ potential of the color-singlet condensates, so the vacua and the flat direction of Sec. V E are the same for every allowed size of these coefficients, zero included.

(vi) *Flavor and contact interactions.* At a cutoff of about 1 TeV, generic $S_3$-invariant couplings that are off diagonal in the generations give tree-level flavor-changing neutral currents, and four-quark contact interactions of the condensing size;

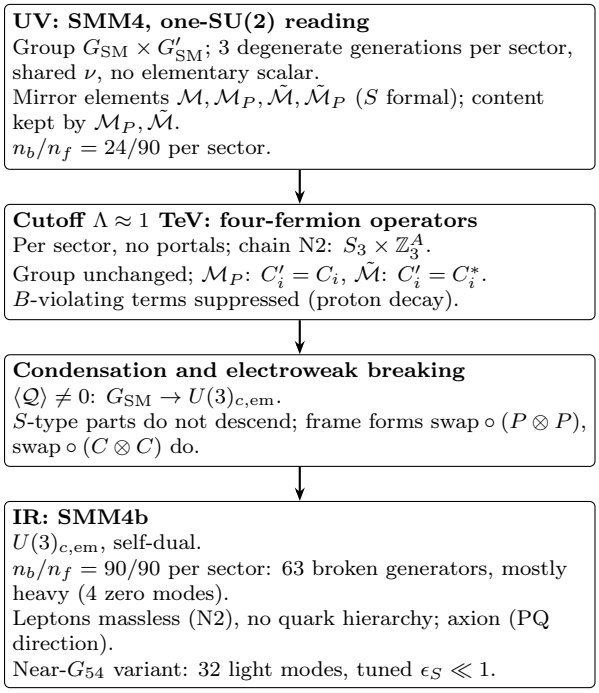



FIG. 1. Schematic of the flow from SMM4 to SMM4b along chain N2: at each stage the gauge group, the surviving mirror elements and the counts $n_b/n_f$ of degrees of freedom per sector. The strong–weak ($S$) part of the mirror elements, a formal map throughout, is lost at condensation; only the classical frame forms of $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ act in the infrared.

both violate the bounds from neutral-meson mixing [40] and from dijet angular distributions at the LHC [41, 42], which apply to any completion that restores the light quarks. We do not compute them.

The chains are therefore a dynamical existence proof for the pattern $G_{\rm SM} \to U(3)_{c,\rm em}$ by quark condensates, with the mirror maps acting as described in Sec. VI, not a realistic model. The numbers of Sec. VI E that use the real-world light-quark spectrum (the QCD vacuum energy, the mirror neutron mass, the bound on $\bar\theta$) refer to the two-doublet model, not to the chains. Figure 1 summarizes the flow.

### C. Matching along the flow

't Hooft anomaly matching [43] holds for the global symmetries common to the ultraviolet and the infrared of chain N2, $U(1)_{B-L} \times U(1)_{B'-L'}$ together with $\mathbb{Z}_3^A \times \mathbb{Z}_3^{A'}$ and $S_3 \times S'_3$. In the ultraviolet $(B-L)^3 = (B-L)\,\mathrm{grav}^2 = -3$ and $(B'-L')^3 = (B'-L')\,\mathrm{grav}^2 = +3$, because the shared $\nu^c$ sits in the mirror doublet; all mixed coefficients vanish. In the 63 vacua all quark masses are vector-like under the unbroken symmetries, and the anomalies are carried by the 18 massless lepton components (in chain N2 the charged leptons and the neutrinos are massless). A $\nu_L \nu^c$ mass, which would be a portal, would gap them and leave only the total $B - L$ matched: with $B - L$ and $B' - L'$ separate, the neutrinos must stay massless. The matching thus assumes, as chain N2 provides, that $B - L$ is conserved in each sector up to its gravitational anomaly; a per-sector Majorana mass would break it. The fermions of the infrared are the elementary fermions of the ultraviolet, so their counts agree exactly. The bosons grow from $n_b = 24$ (12 massless gauge fields) to $90 = 24 + 3 + 63$ (or $96 = 24 + 72$ as distinct states), and the whole excess is composite. The supertrace goes from $n_b - n_f = -66$ to 0: the pseudo-supersymmetric balance is a property of the infrared only. As a consistency check, the $a$-theorem [44] holds for free-field values, although the ultraviolet is not a conformal theory ($U(1)_Y$ is not asymptotically free) and QCD confines in the infrared: in units of a real scalar, $a_{\rm UV} = 991.5$, against at most 670.5 with all 63 pNGBs light.

The vacuum angle behaves as follows. Above $\Lambda$ both $\theta_3$ and $\theta_2$ can be rotated away by anomalous $U(1)$ rotations. At $\Lambda$ the operators of chain N2 break $U_A(1)$ to $\mathbb{Z}_3^A$, but the classical $U(1)$ with charges $X = (1, 1, 1, -3, -3)$ on $(Q, u^c, d^c, L, e^c)$ survives, with an $SU(3)_c^2$ anomaly and no $SU(2)^2$ anomaly, so $\theta_3$ stays unphysical. At condensation $X$ is broken spontaneously; its would-be Goldstone boson, the direction PQ of Sec. V E ($X$ up to the unbroken $B - L$), gets its mass from the anomaly alone. With $\bar\theta = \theta_3 - s_\theta \arg\det \mathcal{Q}$, where $s_\theta = \pm 1$ is the sign with which an anomalous rotation shifts $\theta_3$ (both signs give the same conclusions), the only $X$-breaking term is $-E_\theta \cos\bar\theta$, and $\bar\theta$ relaxes to 0 for $E_\theta > 0$ (to $\pi$ otherwise). The $\mathbb{Z}_3^A$ that forbids $O_{quqd}$ is what keeps this cancellation: with $O_{quqd}$, or with $O_{ledq}$ and $O_{lequ}$ together, no classical $U(1)$ shifts $\theta_3$, and $\bar\theta \neq 0$ generically. The physical meaning of this mode is discussed in Sec. V E.

### D. The condensate-sharing alternative on the same flow

The shared objects of the condensate-sharing alternative must come out of the same per-sector dynamics, and they do so only as vacuum values: the scalars of S1 and S2a arise in the left–right channels above, the currents of S2b never condense, and identifying the fields would need portals (Sec. III D). The 45 pNGBs of $G_{54}$ occur exactly in the 63 class of chain N2, but there they are not a light multiplet; in the near-$G_{54}$ chain the 36 light and zero modes approach the orbit of $U_f(6)_L$, and the "+9" of the count corresponds to the 10 heavy directions. Sharing the vacuum values does not require equal couplings in the two sectors, while sharing the fluctuations as well, as the count of the shared modes assumes, holds if and only if $K'_t = K_t$ on every shared type, with $K_t = G_t^{-1}$ as in Sec. V A; the Jarlskog invariant [45] vanishes in all 55 vacua of the 45 class in the main sample.

### E. An exact flat direction: the axion of the chains

The mean-field potential of chains N2 and near-$G_{54}$ is invariant under separate phase rotations of the up and down condensates. Besides hypercharge, the common rotation $u_R, d_R \to e^{i\alpha} u_R, d_R$ is an exact zero mode of the 72-dimensional Hessian; it is not a gauge direction, and it has an $SU(3)_c^2$ anomaly ($\arg\det(\mathcal{Q}_u \mathcal{Q}_d)$ shifts by $-6\alpha$). In the full Lagrangian of chain N2 the $B$-violating operator $d^c u^c u^c e^c$ breaks this quark rotation unless $e^c$ rotates as well, with charge 3. The exact symmetry is PQ $= (0, -1, -1, 0, 3) = -\frac{1}{2}X + \frac{3}{2}(B - L)$ on $(Q, u^c, d^c, L, e^c)$, which agrees with the quark rotation on the quarks and, up to the unbroken $B - L$, is the $X$ of Sec. V C; we call this direction PQ. Both $B$-violating operators that $\mathbb{Z}_3^A$ allows, $QQQL$ and $d^c u^c u^c e^c$, are PQ neutral, so PQ is a classical symmetry for every value of their coefficients, including the tiny values that proton decay requires (Sec. V B) and zero. Only its charges on the leptons depend on which of them is present: with both absent, or with $U(1)_B$ imposed, it can be taken as the pure quark rotation $(0, -1, -1, 0, 0)$. Its action on the condensates, and hence the axion, is the same in every case. Its only potential is the $\bar{\theta}$-dependent QCD vacuum energy, which relaxes $\bar{\theta}$ to zero in each sector: $\bar{\theta}$ is not a parameter of a sector at leading $N_c$. The price is a pseudoscalar, an axion of the Weinberg–Wilczek type [46–48] with decay constant $f \sim v$. At leading order its mass is $m_a = \sqrt{\chi}\, N_g (x + 1/x)/v$, with $x$ the ratio of the up-type and down-type contributions to $v$ and $\chi$ the topological susceptibility. With the light-quark susceptibility of the real world, $\chi_{\rm top} = (75.7\ {\rm MeV})^4$ [49], this gives $m_a = 140$–$163$ keV on the vacua of both chains. In the chains' own vacua, however, all quarks are heavy (Sec. V B), and the relevant $\chi$ is that of the theory without light quarks: the real-world pure Yang–Mills value $\chi_{\rm YM} = (185\ {\rm MeV})^4$ [50] gives 0.83–0.97 MeV, and it is only a lower reference. Running the strong coupling down through the chains' quark thresholds, with the real-world coupling above all of them, gives a confinement scale of a few GeV, $\chi^{1/4} \approx 2$–$4$ GeV, and $m_a \sim 0.1$–$0.3$ GeV, an order-of-magnitude estimate. The keV value applies only if the light-quark spectrum were restored by a mechanism that the chains do not contain. The original Weinberg–Wilczek axion, with $f$ of the order of the electroweak scale and the light quarks of the real world, is excluded experimentally [51]; in a sector with the chains' heavy quarks its couplings and bounds are not computed here. Within chains N2 and near-$G_{54}$ as formulated the axion is an open problem, which we state rather than solve. The variant of Sec. V B with massive charged leptons does not remove it. There PQ is broken explicitly, by $O_{ledq}$ and $O_{duql}$, but ${\rm PQ}_U = (0, -1, -1, -2, 1)$ is an exact flat direction; it agrees with PQ on the quarks, so the analysis and the mass formula above carry over. Its lowest $B-L$-neutral lifting operators have dimension 18, against 10 in chain N2, and the lowest one with a vacuum value is again $(O_{quqd})^3$.

What would lift the flat direction is constrained by the symmetries of the chain. In chain N2 the lowest $X$-violating operators have dimension 7 (for example $d^c d^c d^c\, \partial e^{c\dagger}$) and the $\Delta B = 2$ operator $(u^c d^c d^c)^2$ is allowed at dimension 9, but all of these violate $B - L$ and have no vacuum value. The lowest $B - L$-neutral lifting operators have dimension 10, and the lowest one with a vacuum value at leading $N_c$, without $B$-violating insertions, is $(O_{quqd})^3$, of dimension 18. Lifts below dimension 18 need insertions of the $B$-violating coefficients (three or four for those of dimension 10, which then carry $(C\Lambda^2)^3 < 10^{-75}$ at the proton-decay bound), so in practice the lowest lift is again $(O_{quqd})^3$. Relaxing $\mathbb{Z}_3^A$ allows $O_{quqd}$ at dimension 6, which lifts the axion at first order. None of the 207 inequivalent anomaly-free flavor-universal $\mathbb{Z}_N$ ($N \le 12$) that forbid $O_{quqd}$ does better: each keeps the axion, with $(O_{quqd})^3$ as the lowest lifting operator. Lifting it costs the relaxation of $\bar{\theta}$. With an explicit breaking $-{\rm Re}[c\,\mathcal{O}]$, where $\mathcal{O}$ is the mean-field image of $O_{quqd}$ or of $(O_{quqd})^3$, an axion mass of 10 GeV (a benchmark beyond the reach of searches for visible axions, not a computed bound) needs a breaking amplitude at least $1.7 \times 10^{19}$ times the largest one compatible with $|\bar{\theta}| < 7.5 \times 10^{-11}$, the bound that follows from the limit on the neutron electric dipole moment [52] with the QCD sum-rule estimate of its $\bar{\theta}$ dependence [53]. For a dominant breaking, $\bar{\theta} = \theta_0 - 3\delta/q$, where $\delta$ is the phase of $c$ and $q = 1$ or 3 for $O_{quqd}$ or $(O_{quqd})^3$: $\bar{\theta}$ is again a free parameter of natural size $O(1)$. In many vacua a diagonal $\mathbb{Z}_3 \subset S_3 \times \mathbb{Z}_3^A$ survives, and there $O_{quqd}$ lifts the axion only at third order in $c$. Thus, within these chains, removing the axion reintroduces the strong CP problem; the effect of the breaking on the vacua and on the mirror maps is described in Sec. VI E and App. F.

## VI. MIRROR MAPS IN SMM4B AND THEIR DESCENT

In the frames $D$ and $D'$ of Paper I the mirror sector is electric, and the content-preserving mirror elements act classically: $\mathcal{M}_P$ as swap$\circ(P \otimes P)$ and $\tilde{\mathcal{M}}$ as swap$\circ(C \otimes C)$ (Sec. III B). We now ask how these maps act on the condensates, the pNGBs and the vacua of SMM4b, what remains of their strong–weak part below the condensation scale, and whether the vacua respect them. As before, $S$ is a formal map and all statements about $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ are statements about their frame forms.

### A. Condensate, flavor group and vacuum

In the neutrino-sharing scenario, with the mirror condensate $\mathcal{Q}'_{ij} = \bar{q}'_{Lj} q'_{Ri}$ built from right-handed mirror doublets, $\mathcal{M}_P$ acts as $\mathcal{Q} \to \mathcal{Q}'(Px)$ and $\tilde{\mathcal{M}}$ as $\mathcal{Q} \to \mathcal{Q}'^*$. On the flavor group, $\mathcal{M}_P$ maps $U_f(6)_L$ to $U_f(6)'_R$ and $U_f(6)_R$ to $U_f(6)'_L$, as it must, since the mirror doublets are right-handed. The $SU_c(3)^2$ anomaly of every flavor

generator changes sign under $\mathcal{M}_P$ and is kept by $\tilde{\mathcal{M}}$, so $U_A(1)$ maps to $U_A(1)'$, and $\bar{\theta} = \theta - s_\theta \arg\det\mathcal{Q}$ (with $s_\theta$ as in Sec. V C) maps to $-\bar{\theta}'$ under $\mathcal{M}_P$ and to $+\bar{\theta}'$ under $\tilde{\mathcal{M}}$; the $\eta'$ relaxation of Sec. V C is mapped consistently. For equal vacuum values, both maps send the unbroken group, the 63 pNGBs, the $\eta'$, the radial modes and the eaten modes onto those of the mirror. The image of the ordinary vacuum is a mirror vacuum if and only if the up and down spectra and the CKM invariants agree, with $J' = J$ for $\mathcal{M}_P$ and $J' = -J$ for $\tilde{\mathcal{M}}$. This is the condensate form of the statement for elementary Yukawa couplings, where invariance under $\mathcal{M}_P$ fixes the mirror Yukawa couplings and requires $m'_f = m_f$, $J' = J$ (in the mirror labeling) and $\bar{\theta}' = -\bar{\theta}$.

On the two-sector content with shared neutrinos only $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ preserve the chiral fermions; all eight frame forms preserve the gauge fields and the condensates, and after electroweak breaking all eight preserve everything. So the pNGB sector does not distinguish the mirror elements; the chiral fermions do. $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ map gauge pseudo-multiplets to gauge pseudo-multiplets and chiral ones to chiral ones, exchanging $\nu_L \leftrightarrow \nu^c$. The exact flavor symmetry of the two sectors is $U(3)^{10}$. Of the flavor-universal $U(1)$'s, $Y$, $Y'$, $B-L$ and $B'-L'$ are free of ABJ anomalies, but as exact symmetries in gravitational backgrounds only $Y$, $Y'$ and the total $(B-L)+(B'-L')$ survive, because each sector's $B-L$ has gravitational coefficient $\mp 3$; $\mathcal{M}_P$ maps $B-L$ to $B'-L'$ and $\tilde{\mathcal{M}}$ to $-(B'-L')$. Below the electroweak scale all eight frame forms preserve the gauge content, and the four-fermion channels and condensate masses select none of them; the weak Fermi currents, at order $G_F$, keep exactly one form of $\mathcal{M}_P$ and one of $\tilde{\mathcal{M}}$. In the vacua of chain N2 with 61 or 63 pNGBs, $J = 0$, so the two conditions $J' = \pm J$ coincide. In the 65 class $J \neq 0$ does occur (7 of 109 vacua of the CP sample of App. F 1, two of them from real couplings, i.e. spontaneous CP violation), and there the vacuum conditions of $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ differ. Some vacua also have exactly degenerate images with different CKM invariants, so that the mirror sector may sit in a vacuum that is not the image of the ordinary one; we return to this in Sec. VI E.

### B. Where the strong–weak part is lost

A mirror element of the ultraviolet induces one in the infrared if and only if it preserves the charges whose magnetic part lies along the light directions of the Higgs phase. For $G_{\rm SM}$ broken by the condensates of Eq. (8) this fails for every $S$-type element in the original frame: none of $\mathcal{M}$, $\mathcal{M}_P$, $\tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}_P$ descends, for any of the 24 root-datum isomorphisms. In the frames where the mirror condensate is electric, all four descend, as their classical frame forms; for $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ these are the infrared maps of Sec. VI A. An $S$ on $\tau_3$ never descends. The infrared group $U(3)_{c,\rm em}$ has its own $S$-type maps, $\tau_3 \to -1/\tau_3$, $\tau_Q \to -1/(9\tau_Q)$, but they are images of no ultraviolet element. The strong–weak part is lost at the condensation and electroweak breaking step, not at the cutoff $\Lambda$, where the gauge group is unchanged. The coupling relations follow the same pattern. In the original frame the elements relate $\tau'_k = -1/\tau_k$ ($\mathcal{M}$, $\tilde{\mathcal{M}}$) or $\tau'_k = 1/\bar{\tau}_k$ ($\mathcal{M}_P$, $\tilde{\mathcal{M}}_P$), i.e. $g_k g'_k = 4\pi$ and $g_Y g'_Y = 12\pi$ at $\theta = 0$; in the frames $D$ and $D'$ they relate $g' = g$ with $\theta' = \mp\theta$, and the four-fermion couplings by $C'_i = C_i$ ($\mathcal{M}_P$) or $C'_i = C_i^*$ ($\tilde{\mathcal{M}}$). The frame relations are invariant under the renormalization group, since both sectors have the same $b_0$; the original-frame relation $gg' = 4\pi$ at all scales would need $b_0 = 0$. At the condensation scale only the frame relations descend, to $e' = e$, $\theta'_{\rm em} = \mp\theta_{\rm em}$, $\bar{\theta}' = \mp\bar{\theta}$ and equal masses, with

$$\tau_Q = \frac{\hat{\tau}_Y}{6} + \frac{\tau_2}{2} = \frac{\theta_{\rm em}}{2\pi} + \frac{2\pi i}{e^2}, \qquad \frac{1}{e^2} = \frac{1}{g_2^2} + \frac{1}{g_Y^2}. \quad (15)$$

The naive restriction of the original-frame relation is not a function of $e$, and the relation $ee' = 6\pi$ of the infrared $S$ is not induced from the ultraviolet.

Since $S$ was a formal map from the outset, the outcome in the infrared is the mirror parity swap $\circ\,(P \otimes P)$ and its CP partner swap $\circ\,(C \otimes C)$, the exchange symmetries of conventional mirror matter [1–3]; the results that depend neither on $S$ nor on the frame are those listed in Sec. I.

The same holds in the condensate-sharing alternative, for both chains and all readings: the condensates $\mathcal{H}^0_{u,d}$ define the same light directions, so the descent is the same, and the identification $\mathcal{H}^0 = \zeta\mathcal{H}'^0$, which no $S$-type element respects at the level of $G_{\rm SM} \times G'_{\rm SM}$, is invisible in the infrared.

### C. The condensate-sharing alternative

On the vacua of both chains, $\mathcal{M}_P$ acts as $\mathcal{Q}' = \mathcal{Q}$, $K' = K$ and $\tilde{\mathcal{M}}$ as $\mathcal{Q}' = \mathcal{Q}^*$, $K' = K^*$, and both send the pNGB spaces, the eaten and radial modes and the shared sets of S1 and S2a onto the mirror's own; the $U_f(6)$ of the left-handed quarks maps onto the flavor group of the mirror doublets, which are right-handed. $\mathcal{M}_P$, with the trivial flavor lift, is compatible with sharing both the vacuum values and their fluctuations. $\tilde{\mathcal{M}}$ is compatible with sharing the vacuum values, through a flavor rotation in the commutant $U(3)^3$ that maps $\mathcal{Q}^*$ back to $\mathcal{Q}$ (the commutant lift of $\tilde{\mathcal{M}}$), but in general not the fluctuations: the rotated couplings differ from $K$ by a relative amount of order one in chain N2 and of order $\epsilon_S$ in the near-$G_{54}$ chain.

### D. A lift with $H \to -H'$

Reference [10] identifies the sectors with the sign rule $\psi_L \to -\psi'_L$, $\psi_R \to \psi'_R$, $H \to -H'$. The corresponding

parity-type lift is

$$\psi_L \to -\gamma^0\psi'_R, \qquad \psi_R \to \gamma^0\psi'_L, \qquad \phi_{u,d} \to -\phi'_{u,d}, \tag{16}$$

with gauge fields mapped as under $\mathcal{M}_P$, and with the mirror leg either (i) with the minus sign on every left-handed mirror field, or (ii) with the minus sign on the mirror doublets only. In frame $D$ both options map every Yukawa and auxiliary-field coupling and every four-fermion operator of the chains onto its mirror partner with $y' = y$ and $K' = K$, exactly as $\mathcal{M}_P$ does; the sign of the Higgs fields is then forced. Option (ii) is an involution, $\mathcal{X}_{\rm ii}^2 = 1$, and is $\mathcal{M}_P$ composed with the central element $z = (-1_2, -1'_2)$ of $G_{\rm SM} \times G'_{\rm SM}$: $\mathcal{X}_{\rm ii} = \mathcal{M}_P \circ z$, i.e. $\mathcal{M}_P$ up to a gauge transformation, in every global form. Option (i) squares to $(-1)^F$ and is $\mathcal{X}_{\rm i} = \mathcal{X}_{\rm ii} \circ (-1)^{F'}$; since no central element acts as $(-1)^{F'}$, it is a distinct lift: up to $z$ it is the lift $\mathcal{M}_P \circ (-1)^{F'}$ of Paper I (its Sec. V C), which squares to $-1$ on the lepton doublets. With the sign $\nu'_L = -\nu_L$ of Ref. [10] for the shared neutrinos, only option (ii) is defined on the shared content; it is the map $\mathcal{M}_P$ of Sec. III B with doublet phases $-1$, and squares to $+1$ on the shared pair. On shared condensates both options act as $\mathcal{H} \to -\mathcal{H}'$, so a vacuum on the sharing surface $\mathcal{H} = \zeta\mathcal{H}'$ is invariant for $\zeta = -1$ (for $\zeta = +1$ under the uniform $\mathcal{M}_P$). Option (ii) allows exactly the same portal operators as $\mathcal{M}_P$, operator by operator, and gives the same $n$–$n'$ mass family as in Paper I; option (i) differs only in the fermionic portals, with the same $n$–$n'$ masses and mixing angle. The sign $H \to -H'$ is invisible in every gauge-invariant operator, and neither option has a perturbative or instanton-vertex anomaly. The chirality-keeping map of Ref. [10], $T$, is $\tilde{\mathcal{M}}$ up to the gauge element $z$ on the copy content; option (i) is $T$ followed by parity in both sectors, and on the two-sector content with the shared neutrinos of Sec. III B and the neutrino sign of Ref. [10] the lift is option (ii). The lift and $T$ thus differ by a CP-type transformation in each sector, which is the difference between $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ in Paper I. In short, on this content the duality operator with $H \to -H'$ is option (ii), which is $\mathcal{M}_P$ up to a gauge transformation.

### E. Spontaneous mirror breaking and CP

The $n$–$n'$ oscillations of Ref. [9] need a small asymmetry $\varepsilon = (v' - v)/v$ of the vacuum values. This asymmetry sets only the splitting of $n$ and $n'$; the mixing $\delta_m$ is a cross-sector effect, a portal or the non-perturbative origin discussed in Paper I, outside the present setting, and we take it as an input (App. F 2). Through the relation $\Delta m_n = \kappa_n m_n \varepsilon$ of Paper I (its Sec. V B), with $\kappa_n = d\ln m_n/d\ln v \simeq 0.30$ (written $\kappa$ in Paper I), the splitting $\Delta \simeq$ 3–12 $\mu$eV of Ref. [9] needs $|\varepsilon| \simeq 1.1$–$4.2\times10^{-14}$; the sign of Ref. [9], with $n'$ heavier, is $v' > v$. The relation assumes that all quark masses scale with $v$. Asymmetric mirror vacua, $v' \neq v$, have been studied with a mirror $\mathbb{Z}_2$ that is broken softly or explicitly, often with $v' \gg v$ [23, 54, 55], and as an asymmetric vacuum of an exactly $\mathbb{Z}_2$-symmetric Higgs potential in which the two Higgs doublets couple [56, 57]. Here, with the mirror elements exact, no portals and equal couplings, $K' = K$, the two sectors have the same potential and no coupling between them; we ask whether they can nevertheless settle in different vacua. The answer is yes, but not by itself at the size required. (In the condensate-sharing alternative a shared vacuum value forces $\varepsilon = 0$ on the shared quark type, Sec. III D; we consider here the neutrino-sharing scenario and the two-doublet model.)

By the two-doublet model we mean SMM4 with elementary doublets $\phi_u, \phi_d$ in each sector, extended to three generations with the type-II Yukawa couplings $Qu^c\phi_u$, $Qd^c\phi_d$, $Le^c\phi_d$ and the real-world quark spectrum, and with the most general real tree-level potential, Eq. (F4), with mass terms $m_{11}^2$, $m_{22}^2$, $m_{12}^2$ and quartic couplings $\lambda_1, \ldots, \lambda_7$. Its $\mathbb{Z}_2$, $\phi_u \to -\phi_u$, which $m_{12}^2$ breaks softly, acts within one sector and is not the mirror $\mathbb{Z}_2$. Two inequivalent local minima of this potential coexist in part of the parameter space, never more than two, and never a charge- or CP-breaking minimum next to a neutral one, in line with general results on two-doublet potentials [58–61]. In the softly broken $\mathbb{Z}_2$ case, a second minimum with the opposite sign of $v_u v_d$ remains a strict local minimum for small $m_{12}^2$, with $\varepsilon$ linear in $m_{12}^2$; it flips the sign of $m_u m_d$, so without an axion a sector in it has $\bar\theta + \pi$. At the representative point of App. F 1, $\varepsilon \simeq -1.10\, m_{12}^2/|m_{11}^2|$ at tree level, so the window needs $|m_{12}^2/m_{11}^2|$ of order $10^{-14}$, with the ordinary sector in the false vacuum for $v' > v$ and the mirror for $v' < v$ (App. F 2). These values hold only if $\bar\theta$ is relaxed in each sector; the QCD vacuum energy discussed below changes them otherwise.

The two sectors are forced to have equal vacua only where the per-sector minimum is unique, as for $U(3)^5$ operators; otherwise they have equal vacua when both sit in the same minimum. Wherever competing local minima exist, in the two-doublet model and in chains N2 and near-$G_{54}$, the separable two-sector potential has mixed local minima, exactly degenerate with their exchange images, and the Lagrangian does not choose between them. A configuration with $\varepsilon \neq 0$ then needs a cosmological history that selects a single minimum in each sector over the observable universe. If the electroweak transition of a sector occurs after inflation, regions that settle in different minima are separated by domain walls [62–64]. With the bias that the window needs, such walls collapse before the QCD transition, in both sectors toward the minimum favored by the tree-level bias at the electroweak transition [65, 66], which removes the asymmetric configuration instead of leaving it. At the electroweak transition only the tree-level bias exists, and it has the sign of $m_{12}^2$. In every tuned case of App. F 2 except tree-level tuning with $v' < v$, $m_{12}^2 < 0$ and the walls of both sectors collapse into the mirror's minimum, so the history must keep our own sector out of the minimum favored at its electroweak transition: for example, low-scale inflation (Hubble rate

well below the electroweak scale) and a transition that precedes inflation, with a reheating temperature below the electroweak transition in our sector [67, 68], with consequences for baryogenesis. An asymmetric reheating of the two sectors [54] alone does not suffice. Exactly degenerate vacua, such as the CP images and the relative $S_3$ images below, would instead form stable walls, which cosmology excludes [62]; a small lifting of the degeneracy only turns them into biased walls, which collapse to the lower image in both sectors, so these routes too need a history of the same kind. We do not model the history itself. Couplings between the sectors would not produce a small asymmetry either. With one Higgs doublet per sector, an exact symmetric quartic portal $\lambda_p|\phi|^2|\phi'|^2$ (excluded here) has only the symmetric vacuum for $|\lambda_p| < 2\lambda$ and gives $\varepsilon = -1$, one sector unbroken, for $\lambda_p > 2\lambda$ ($\lambda_p = 2\lambda$ is a flat direction). Gravity would shift $\varepsilon$ by at most about $10^{-33}$, and so would a loop of the shared neutrinos with Yukawa couplings of the size of a Dirac neutrino mass.

Untuned, the asymmetry is large: $|\varepsilon| \geq 2.7 \times 10^{-3}$ in the targeted scan of the two-doublet model and $|\varepsilon_{\rm rms}| \geq 3.6\times10^{-4}$ in the competitor search of chain N2 (App. F 1, where $\varepsilon_{\rm rms}$, the change of the root-mean-square quark mass, is defined). For the chains the window is only an order of magnitude for a generic asymmetry of the vacuum values: a competing condensate minimum changes the quark masses non-uniformly, and the chains have no light quarks (Sec. V B), so the universal scaling behind $\kappa_n$ does not hold there. Reaching $10^{-14}$ needs a tuned small parameter: in the near-$G_{54}$ chain a splitting $\epsilon_S = 1.3 \times 10^{-13}$–$3.4 \times 10^{-12}$, extrapolated linearly to $|\varepsilon_{\rm rms}| = 10^{-14}$ from the scaling set, measured at $\epsilon_S \geq 10^{-3}$ (the ratios $|\varepsilon_{\rm rms}|/\epsilon_S$ of the 48 mixed configurations of the competitor search, App. F 3, would give a wider range); a cusp tuning of two merging minima; or the soft breaking of the $\mathbb{Z}_2$ of the two-doublet potential above.

The $\bar\theta$-dependent QCD vacuum energy lifts the degeneracy at $m_{12}^2 = 0$. At leading order in chiral perturbation theory [69–71], a mirror at $\bar\theta' = \bar\theta + \pi$ carries the extra energy $\Delta E = 2\Sigma_0 m_u$, with $\Sigma_0$ the quark condensate, which shifts $\varepsilon$ by about $-10^{-13}$ (App. F 2), of the order of the window, but is too small, by a factor of order $10^{12}$ against the electroweak barrier, to select the vacua. With this shift both sign branches of the window need $m_{12}^2/|m_{11}^2| \approx -10^{-13}$ (App. F 2). This tuning, however, gives no $n$–$n'$ window, because the shift exists only for a mirror at $\bar\theta' = \bar\theta + \pi$, which is very different: the physical $m_u/m_d = 0.46$ lies below the window of Dashen's phenomenon [72], in which CP breaks spontaneously at $\theta = \pi$ (App. F 2; see Refs. [73, 74] for three flavors), so CP is conserved, but the charged mirror pions weigh 82 MeV and $m_{n'} - m_n = -29$ MeV at leading order, (2.5–9.8) $\times 10^{12}$ times the splitting of Ref. [9]. The soft-$\mathbb{Z}_2$ route to $\varepsilon \sim 10^{-14}$ therefore gives no observable $n$–$n'$ oscillation unless $\bar\theta$ is relaxed in each sector separately. In chains N2 and near-$G_{54}$ there is no such branch: the flat direction of Sec. V E relaxes $\bar\theta$ to zero in each sector, so there is no $\bar\theta' = \pi$ vacuum and no QCD-induced $\varepsilon$, and the QCD energy is far too small to reorder their vacua.

In the chains themselves, the competitor search finds competing condensate minima in at least 44 of 240 quark types of chain N2 and 41 of 180 of the near-$G_{54}$ chain, and in none of the $U(3)^5$ controls. A mirror sector in a competing minimum, with the ordinary one in the global minimum, is often a local minimum of the 72-dimensional potential; the stability depends on the relative $S_3$ image of the up and down condensates, which costs no energy. None of these configurations has $J \neq 0$, and their asymmetries are given in App. F 3, with the scheme dependence of $\varepsilon_Z$ and $\varepsilon_W$.

Spontaneous breaking of CP offers a second route, which breaks $\mathcal{M}_P$ and CP together while keeping their product. If the vacuum of a sector breaks CP spontaneously and the sector Lagrangian is CP invariant, the mirror can sit in the CP image of the ordinary vacuum: the configuration is exactly degenerate, has $v' = v$ and equal masses, and reverses every CP-odd quantity relative to the $\mathcal{M}_P$ image. It breaks $\mathcal{M}_P$ and keeps $\tilde{\mathcal{M}} = \mathcal{M}_P \circ CP$. In the soft-$\mathbb{Z}_2$ two-doublet model with a CP-violating global minimum, however, $\bar\theta = 3\xi$ in terms of the relative phase $\xi$ of the doublets, and the bound on the neutron electric dipole moment [52, 53] requires $m_{12}^2$ to be tuned to a relative $4.3 \times 10^{-11}$. Under $\tilde{\mathcal{M}}$ the $n$–$n'$ mass family has 4 physical parameters instead of the 3 of $\mathcal{M}_P$, the new one a phase, but every mass eigenstate still has equal $n$ and $n'$ content: $\theta = \pi/4$ as for $\mathcal{M}_P$, which is the degenerate regime of Paper I rather than the one of Ref. [9]. In the chains the couplings are complex, so the CP image is not degenerate; but the six relative $S_3$ images of a vacuum are, and exactly four vacua of the main sample have $J \neq 0$ in some images and $J = 0$ in others (in both chains; one of them is a saddle). A sector with $J \neq 0$ and a mirror with $J' = 0$ therefore cost no energy at leading $N_c$, with equal quark masses, apart from the domain walls discussed above; the resulting asymmetry $\varepsilon_W = v'_W/v_W - 1$ of the vacuum value $v_W$ that sets the $W$ mass is of order $10^{-5}$ to $10^{-3}$ in the three stable cases (App. F 3). Here $J = 0$ means $|J| \lesssim 10^{-13}$, the numerical precision of the couplings (for the mixed configurations, the criterion stated there). Finally, a single axion coupled to both sectors, as in Ref. [75], with $\bar\theta'_{\rm eff} = \bar\theta_{\rm eff} + \pi$ settles at $|\bar\theta| = \pi/2$ in both sectors, far above the electric-dipole bound.

If the axion of Sec. V E is lifted by an explicit $O_{quqd}$-type term with coefficient $c$, $\mathcal{M}_P$ and both lift options give $c' = c$, hence $v' = v$ and $\bar\theta' = -\bar\theta$; $\tilde{\mathcal{M}}$ gives $c' = c^*$ and $\bar\theta' = +\bar\theta$. The breaking splits the degenerate pair and makes $J \neq 0$ in many vacua where it vanished before, and there the commutant lift of $\tilde{\mathcal{M}}$ of Sec. VI C fails. A mismatch $|c'| \neq |c|$ alone would give an asymmetry $\varepsilon = (d\ln v/d\ln|c|)\,\delta|c|/|c|$, and an asymmetry of the size of the window of Ref. [9], as an order-of-magnitude target for these vacua, would need $\delta|c|/|c|$ of about $10^{-11}$ to $10^{-8}$.

## VII. READINGS WITH GAUGE GROUP $G_{\rm SM} \times U(2)$

In the one-SU(2) reading of Sec. III, $SU_I(2)$ and $SU_L(2)$ are two halves of one gauge group and each sector is exactly $G_{\rm SM}$. For comparison we now consider the alternatives in which $SU_I(2)$ and the $U_{BL}$ of Ref. [10] are gauged as separate factors, so that each sector has, abstractly, the gauge group $G_{\rm SM} \times U(2) \cong S(U(2) \times U(3) \times U(2))$ of Eq. (2), with $U(2) = (U_{BL} \times SU_I(2))/\mathbb{Z}_2$. They are related to left–right symmetric extensions with gauged $B - L$ [76–79], but here $U_{BL}$ acts only on left-handed doublets. We summarize the results; the anomaly coefficients, search boxes and minimal sets are in App. B, the conventions for $U_{BL}$ and its kinetic and $\theta$ mixing with $U(1)_Y$ in App. A.

### A. Definitions

We write left-handed Weyl multiplets as $(\mathbf{c}, \mathbf{d}_L, \mathbf{d}_I)_{Y,q}$ under $SU_c(3) \times SU_L(2) \times SU_I(2)$, with hypercharge $Y$ and $U_{BL}$ charge $q$. In each reading $U_{BL}$ acts with integer charges $q_Q$ and $q_L$, on the center circle of $U(2)$, on the left-handed doublets $Q$ and $L$ only, while $u^c, d^c, e^c, \nu^c$ are neutral. The parity class of the reduced charges $(q_Q, q_L)/\gcd(q_Q, q_L)$ is (a) both odd, (b) $q_Q$ odd and $q_L$ even, or (c) $q_Q$ even and $q_L$ odd.

*Bidoublet.* $SU_L(2)$ acts on $Q$ and $L$, and $SU_I(2)$ on the left-handed quarks only, so $Q = (\mathbf{3}, \mathbf{2}, \mathbf{2})_{1/6,q_Q}$ is a bidoublet and $L = (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-1/2,q_L}$. A sector then has 22 Weyl components rather than the 16 of $\Lambda(\mathbb{C}^5)$, and $n_b = 32$, $n_f = 44$.

*R2.* $SU_L(2)$ acts on $L$ only and $SU_I(2)$ on $Q$ only, as independent factors: $Q = (\mathbf{3}, \mathbf{1}, \mathbf{2})_{1/6,q_Q}$, $L = (\mathbf{1}, \mathbf{2}, \mathbf{1})_{-1/2,q_L}$. The content is one generation of $\Lambda(\mathbb{C}^5)$.

*Link field.* $SU_L(2)$ acts on $Q$ and $L$, which keep their Standard Model form, and $SU_I(2)$ acts only on a complex scalar $V = (\mathbf{1}, \mathbf{2}, \bar{\mathbf{2}})_{Y_V,q_V}$ and on the composite $V^\dagger Q$; $\langle V \rangle = v\, 1_2$ breaks $SU_L(2) \times SU_I(2)$ to the diagonal. We keep both $Y_V = 0$ (family A) and $Y_V = \frac{1}{2}$ (family B). Per sector $n_f = 32$ and $n_b = 2 \cdot 16 + 8 = 40$.

When the $G_{\rm SM}$-type factor of a splitting $G_{\rm SM} \times U(2)$ is not the literal one, its hypercharge is sheared, $Y' = Y + s\,q/6$; for family A of the link field and for R2 class (b) we use $s = +3$, i.e. $Y' = Y + q/2$. The opposite sign $s = -3$ is equally allowed and only flips the signs of $\chi$ and $\theta_{YX}$.

### B. Self-duality, anomalies and additions

Each reading is Langlands self-dual only in some charge classes (Table V). The bidoublet content is a representation of the literal $G_{\rm SM} \times U(2)$ if and only if it is in class (b), and for $\gcd(q_Q, q_L) = 1$ this group acts faithfully and is self-dual; class (c) gives a further self-dual form, isomorphic to $G_{\rm SM} \times U(2)$, in which the $U(2)$ center charge mixes with hypercharge, and class (a), which contains the $B - L$ values $(1, -3)$, has none. In R2, classes (b) and (c) are self-dual, always with a sheared hypercharge, and class (a) is not. The link field has four self-dual classes, A, A′ (all charges odd), B (the literal product) and B′, all isomorphic to $G_{\rm SM} \times U(2)$; A′ and B′ admit the $B - L$ charges $q = 3(B - L)$ on $Q$ and $L$. The anomaly conditions do not fix the $U_{BL}$ coupling. We fix it by requiring that the S-duality of the $U(2)$ factor act with a single coupling, $\tau \to -1/\tau$; with integer $q$ and the $U(1)$ normalization of App. A this gives

$$e_{BL} = \frac{g_I}{2}, \qquad \tau_{BL} = 2\tau_{U(2)}, \qquad \tau_{U(2)} = \frac{\theta}{2\pi} + \frac{4\pi i}{g_I^2}, \tag{17}$$

so that $\tau_{U(2)} \to -1/\tau_{U(2)}$ acts as $\tau_{BL} \to -4/\tau_{BL}$. On sheared splittings the same criterion requires $Y$–$BL$ kinetic mixing (App. A 1).

Every reading is anomalous, for every choice of charges, and perturbative coefficients depend on the content, not on the global form. The bidoublet carries the anomalies of one extra $(\mathbf{3}, \mathbf{2})_{1/6}$, among them $A_{SU(3)^3} = 2$, and an odd number (seven) of $SU_L(2)$ doublets. R2 splits the Standard Model $SU(2)$ into a lepton and a quark factor, with $A_{SU_L(2)^2 Y} = -\frac{1}{4}$, $A_{SU_I(2)^2 Y} = +\frac{1}{4}$ and Witten anomalies [21] with one and three doublets, which no $U_{BL}$ charge affects. In the link field all pure $G_{\rm SM}$ coefficients vanish, but $A_{SU(3)^2 BL} = q_Q$ and $A_{SU_L(2)^2 BL} = (3q_Q + q_L)/2$ vanish together only at $q_Q = q_L = 0$. Anomaly-free completions therefore need added Weyl fermions, whose minimal numbers within the search boxes of App. B are listed in Table V; every minimal set keeps the self-dual form named. In the two-sector theories with shared neutrinos, fields charged under both sectors lower the minimum by one in R2 and in the link field, but those sets are not invariant under the mirror parity $\sigma$, which acts on additions as $\sigma : (\mathbf{c}, \mathbf{d}_L, \mathbf{d}_I, y, q \,|\, \mathbf{c}', \mathbf{d}'_L, \mathbf{d}'_I, y', q') \mapsto (\bar{\mathbf{c}}', \mathbf{d}'_L, \mathbf{d}'_I, -y', -q' \,|\, \bar{\mathbf{c}}, \mathbf{d}_L, \mathbf{d}_I, -y, -q)$ with $y = 6Y$, and their cross-sector traces such as $\mathrm{tr}(YY') = -4$ generate a divergent $B$–$B'$ kinetic mixing. With $\sigma$-invariance imposed, the minimum equals twice the one-sector value at every point tested, attained by sector-only sets: the mirror parity removes the saving.

### C. The extended mirror map

The mirror elements of Paper I extend to $G_{\rm SM} \times U(2)$ through the block-diagonal root-datum isomorphisms $\Phi = (Fa) \oplus b$, with $Fa$ one of the 24 isomorphisms of $G_{\rm SM}$ and $b \in \{1, P, -1, -P\}$ on $U(2)$, 96 in all. Of the elements $X = \mathrm{swap} \circ (P \otimes P)^p \circ (S_\Phi \otimes S_{\Phi'})$, none preserves the bidoublet matter in the original frame, and every one acts classically in the S-dual frames $D$ and $D'$ of the mirror. The matter-compatible elements are exactly those of the $\mathcal{M}_P$ type and of the $\tilde{\mathcal{M}}$ type, with the twist locked across the two factors, since $Q$ and $L$ carry both $G_{\rm SM}$

TABLE V. The readings of SMM4 per sector: self-dual forms, characteristic anomalies, minimal anomaly-cancelling additions (left-handed Weyl components) and the one-sector self-dual point at $\theta = 0$. The additions hold within the search boxes of App. B (N: $|Y| \le \frac{3}{2}$, $|q| \le 12$; W: $|Y| \le \frac{5}{2}$, $|q| \le 20$; R: $|Y| \le \frac{3}{2}$, $|q| \le 9$), at the point $(q_Q, q_L[, q_V])$ given, in boxes N and W unless a box is named (R2 in box R); "Witten" marks the odd doublet counts 1 and 3; "two sectors" is the theory with shared neutrinos with $\sigma$-invariant additions. All additions keep the self-dual form named.

| Reading, form | Anomalies | One sector | Two sectors | Self-dual point |
|---|---|---|---|---|
| One SU(2), $G_{\rm SM}$ | none | 0 | 0 | $\alpha_2 = 1$, $\alpha_Y = 3$ |
| Bidoublet (b), literal | $A_{SU(3)^3} = 2$, 7 $SU_L(2)$ doublets | 15 at $(1,-6)$, 13 at $(1,0)$ | 30 at $(1,-6)$ (N) | $\alpha_I = 1$, $\alpha_{BL} = \frac{1}{4}$, $\chi = 0$ |
| Bidoublet (c) | as (b) | 13 at $(0, q_L)$ | $2 \times 13$ | $\chi^2 \ge \frac{3}{4}$ |
| R2 (c) | $A_{SU^2_{L,I}Y} = \mp\frac{1}{4}$, Witten | 4 at $(0,1)$ | 8 (R) | $g_Y^2 = 3\pi$, $\chi = -\frac{\sqrt{3}}{2}$ |
| R2 (b), $SU_I(2)$ on $Q$ | as (c) | 12 at $(1,0)$, 15 at $(1,2)$ (R) | 24, 30 (R) | as (c) |
| Link field A, A′ | $A_{SU(3)^2 BL} = q_Q$ | 9 at $(2,-6,1)$ | 18 | $g_Y^2 = 3\pi$, $\chi = -\frac{\sqrt{3}}{2}$ |
| Link field B, B′ | as A | 9 at $(2,-6,1)$ | 18 | B: $g_Y^2 = 12\pi$, $\chi = 0$ |

and $U_{BL}$ charges; the mirror must have the same class, charges and splitting as the ordinary sector. At $\theta = 0$ the fixed loci of $\mathcal{M}$ and $\mathcal{M}_P$ are

$$\begin{aligned} g_3 g_3' = g_2 g_2' = g_I g_I' = 4\pi, \\ g_Y g_Y' = 12\pi, \qquad e_{BL} e_{BL}' = \pi, \end{aligned} \tag{18}$$

with $e_{BL} e_{BL}' = \pi$ rather than the $2\pi$ of an independent $U(1)$, because $U_{BL}$ is not a direct factor of $U(2)$. Under $\mathcal{M}$ and $\mathcal{M}_P$ every ordinary Wilson line goes to a pure mirror 't Hooft line, and on Fix($\mathcal{M}$) all cross-sector kinetic and $\theta$ mixings vanish. In the link-field reading, however, the $S$-type elements never commute with the breaking $\langle V \rangle = v\, 1_2$, at any physical coupling, while the classical mirror parity $X_{\rm cl}$ commutes with it for every coupling. Whether the mirror relation of the ultraviolet descends to the unbroken group $H_0 \cong G_{\rm SM}$ of the link field is analyzed in App. C: imposed exactly, it does not on any single-coupling locus, and it does with order-one physical angles only when the map may include a shift by a period of the angles. On the abelian part the extended elements act as compact-boson T-dualities, with two bosons per cycle and sector, and on generic Hitchin fibers, $\mathfrak{M}_H(G_{\rm SM} \times U(2)) = \mathfrak{M}_H(G_{\rm SM}) \times \mathfrak{M}_H(GL_2)$, as SYZ T-dualities [15, 80, 81]; in both pictures the twist locking is visible, and only the single $U(2)$ coupling is mapped to itself, with $g_I g_I' = 4\pi$.

## VIII. CONCLUSIONS AND OUTLOOK

We have studied the mirror duality of Paper I in the mirror-matter models SMM4 and SMM4b of Ref. [10]. In the one-SU(2) reading each sector of SMM4 is exactly $G_{\rm SM}$, and with shared neutrinos only $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ preserve the content; the balance $n_b = n_f = 90$ of SMM4b is a count of broken generators, not a supersymmetry. Along the four-fermion chains, an existence proof for the symmetry-breaking pattern rather than a realistic model, the strong–weak part of the duality is lost at condensation, so that in the infrared it acts as the mirror parity swap $\circ\, (P \otimes P)$ or swap $\circ\, (C \otimes C)$, and what does not depend on $S$ is the structure listed in Sec. I.

Compared with the setting of Paper I, with one generation per sector and an elementary Higgs doublet, these models change a few of its results. (i) Below the condensation scale only the classical frame forms of the mirror elements survive (Sec. VI B). (ii) The Yukawa couplings become condensates and four-fermion couplings, with $K' = K$ (Sec. VI A); of Paper I's map between the families of the two sectors (its Sec. IV C), only the quark part and the $B - L$ part transfer with shared neutrinos, since $\mathcal{M}_P$ exchanges $\nu_L$ and $\nu^c$. (iii) On the shared neutrinos the defined lift is option (ii) of Sec. VI D, $\mathcal{M}_P$ up to a gauge transformation, with the same portal selection rules and $n$–$n'$ mass family. (iv) With two elementary doublets per sector and type-II Yukawa couplings, the Yukawa-allowed symmetries include a Peccei–Quinn $U(1)$ with mixed coefficients $A_{B,\rm PQ} = -\frac{1}{3}$ and $A_{L,\rm PQ} = +1$ that no shift removes, so the statement that they form no 2-group fails. (v) A shared vacuum value forces $\varepsilon = 0$ on the shared quark type (Sec. III D). (vi) In a formal one-loop extrapolation, $\alpha_Y = 3$ moves from $1.3 \times 10^{41}$ GeV in the Standard Model to $1.2 \times 10^{42}$ GeV in SMM4b, still above $M_P$.

Several problems remain open. First, the physical status of the per-sector $S$, which the $B + L$ anomaly obstructs on the chiral $SU(2)$ and whose self-dual couplings are never reached below $M_P$; the $\mathcal{N} = 2$ model of App. E has exact dualities, but its two-sector form cannot be exchange symmetric for an odd number of generations. Second, the axion of the chains, of the Weinberg–Wilczek type with $f \sim v$ and an estimated mass of 0.1–0.3 GeV, whose removal reintroduces the strong CP problem. Third, the asymmetry $\varepsilon \sim 10^{-14}$ needed for the $n$–$n'$ oscillations of Ref. [9] requires a tuned parameter and a cosmological history that selects a single minimum in each sector; in the two-doublet model the tuned vacuum puts the mirror at $\bar\theta' = \bar\theta + \pi$, with a neutron 29 MeV lighter, unless $\bar\theta$ is relaxed in each sector, and even then the $n$–$n'$ mixing needs a cross-sector effect outside the present setting. Fourth, we have not addressed the loop-level stability of the vacuum, which in the Standard Model is metastable [82]; the neutrino masses; electroweak precision, flavor-changing neutral currents and

contact interactions at the chains' cutoff of about 1 TeV; the origin of the suppression of their baryon-number-violating couplings; the quark mass hierarchy; and the map of the flavor structure, including the CKM phase, between the sectors.

Shared neutral composites were motivated as a link between the sectors through neutral-hadron oscillations; without portal operators they can be shared only as vacuum values, and an amplitude that mixes ordinary and mirror hadrons needs a mechanism beyond this setting, possibly of the non-perturbative or topological kind discussed in Paper I.

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## Appendix A: Conventions

We use the conventions of Paper I [13] (its App. A): the metric $\mathrm{diag}(+1,-1,-1,-1)$, $\epsilon^{0123}=+1$, $\tau=\theta/2\pi+4\pi i/g^2$ for a simple factor with the $\theta$-term $+\theta g^2F^a\tilde F^a/32\pi^2$, $\tilde F^{\mu\nu}=\frac12\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, and $S:\tau\to-1/\tau$ acting on charges $(n_m,n_e)\mapsto(n_m,n_e)S^{-1}$. A $U(1)$ with integer charges and coupling $e$ is normalized with $g^2=2e^2$, i.e. $\tau=\theta/2\pi+2\pi i/e^2$. For hypercharge, $\hat\tau_Y=3\theta_Y/2\pi+12\pi i/g_Y^2$. This appendix adds the conventions for $U_{BL}$ and its mixing with $U(1)_Y$, which only this paper uses.

*Coupling of $U_{BL}$.* With integer $U_{BL}$ charges $q$ on the center circle of $U(2)$ and the field $X_\mu$, the coupling has the sign of the hypercharge coupling,

$$D_\mu=\partial_\mu-ig_YY\,B_\mu-ie_{BL}\,q\,X_\mu, \tag{A1}$$

not the sign of the QED convention.

*Kinetic mixing.* With both field strengths canonically normalized,

$$\mathcal{L}\supset-\tfrac14B_{\mu\nu}B^{\mu\nu}-\tfrac14X_{\mu\nu}X^{\mu\nu}-\tfrac{\chi}{2}B_{\mu\nu}X^{\mu\nu}. \tag{A2}$$

The kinetic energy is positive if and only if $|\chi|<1$ (the eigenvalues of the kinetic matrix are $1\pm\chi$), and $X\to-X$ maps $\chi\to-\chi$.

*$\theta$-terms.* The abelian $\theta$-terms are

$$\begin{aligned}\mathcal{L}\supset{}&\frac{\theta_Yg_Y^2}{32\pi^2}B_{\mu\nu}\tilde B^{\mu\nu}+\frac{\theta_{BL}e_{BL}^2}{16\pi^2}X_{\mu\nu}\tilde X^{\mu\nu}\\&+\frac{\theta_{YX}g_Ye_{BL}}{16\pi^2}B_{\mu\nu}\tilde X^{\mu\nu}.\end{aligned} \tag{A3}$$

The center of $U(2)$ with the single coupling of Eq. (17) has $\theta_{BL}=2\theta$, where $\theta$ is the angle of $\tau_{U(2)}$.

*Abelian coupling matrix.* With $a_B=(g_Y/N_Y)B$ and $a_X=e_{BL}X$, whose charges $N_YY$ and $q$ are integers for $N_Y\in6\mathbb{Z}$, the abelian Lagrangian is $\mathcal{L}_{\rm ab}=-\frac{1}{8\pi}\mathrm{Im}\,\tau_{ij}\,f_i\cdot f_j+\frac{1}{8\pi}\mathrm{Re}\,\tau_{ij}\,f_i\cdot\tilde f_j$, where $f_i=da_i$ and $f\cdot f=f_{\mu\nu}f^{\mu\nu}$, with

$$\begin{aligned}\tau_{BB}&=\frac{N_Y^2\theta_Y}{4\pi}+\frac{2\pi iN_Y^2}{g_Y^2},\qquad\tau_{XX}=\frac{\theta_{BL}}{2\pi}+\frac{2\pi i}{e_{BL}^2},\\\tau_{BX}&=\frac{N_Y\theta_{YX}}{4\pi}+\frac{2\pi iN_Y\chi}{g_Ye_{BL}},\end{aligned} \tag{A4}$$

so that $\tau_{BB}=(N_Y^2/6)\hat\tau_Y$ and $\mathrm{Im}\,\tau>0$ if and only if $|\chi|<1$. We keep $N_Y$ a parameter. For every global form of this paper the lattice of the center torus is $\mathbb{Z}e_Y\oplus\mathbb{Z}e_{BL}$ and its dual $6\mathbb{Z}e^Y\oplus2\mathbb{Z}e^{BL}$, with $(e^Y,e^{BL})$ the dual basis, so $N_Y=6$ is the lattice basis. We write $\tau_z$ for the matrix of Eq. (A4) at $N_Y=6$, for which $\tau_{BB}=6\hat\tau_Y$.

*The S map on the coupling form.* On the Cartan algebra the coupling form is $\mathcal{T}=\mathrm{diag}(\tau_z,\tau_LA_1,\tau_cA_2,\tau_IA_1)$, with $A_r$ the Cartan matrices, and $\mathcal{L}=-\frac{1}{8\pi}\mathrm{Im}\,\mathcal{T}(f,f)+\frac{1}{8\pi}\mathrm{Re}\,\mathcal{T}(f,\tilde f)$. For a frame $g:X_*\to X^*$, a Langlands self-duality mapping coroots to roots,

$$S_g:\ \mathcal{T}\mapsto-g^T\mathcal{T}^{-1}g,\qquad(m,e)\mapsto(g^{-1}e,\,-g^Tm), \tag{A5}$$

which preserves the Dirac pairing $e_1\cdot m_2-e_2\cdot m_1$, the condition $\mathrm{Im}\,\mathcal{T}>0$ and the mass form $Z^\dagger(\mathrm{Im}\,\mathcal{T})^{-1}Z$ of $Z=e+\mathcal{T}m$, with $S^2=1$ on $\mathcal{T}$ and $S^2=C$ on charges for symmetric $g$. For a single $U(1)$ with integer charges ($g=1$) it is $\tau\to-1/\tau$; for a simple factor, $\tau_a\to-1/\tau_a$; for the center of $U(2)$, $\tau_{BL}\to-4/\tau_{BL}$; for $G_{\rm SM}$, $\hat\tau_Y\to-1/\hat\tau_Y$. The literal frames of the bidoublet and of link family B are $g=\mathrm{diag}(6s_Y,2s_B)$ on $\tau_z$, with signs $s_Y,s_B=\pm1$. The rule $\tau\to-\tau^{-1}$ on the matrix of Eq. (A4) is the S-duality of the product $U(1)_Y\times U_{BL}$ with integer charges $(N_YY,q)$; it is not the S-duality of $G_{\rm SM}$, $U(2)$ or the global forms of this paper, for any $N_Y$.

*Splittings and shear.* A splitting $G=H_1\times H_2$ of a global form with $H_1\cong G_{\rm SM}$ and $H_2\cong U(2)$ is given by primitive center directions $c_u=u$ and $c_v=v$ in the $(e_Y,e_{BL})$ basis, together with the non-abelian factors. On $u=(1,s)$, $v=(0,1)$ the hypercharge of the $G_{\rm SM}$ factor is $Y'=Y+s\,q/6$. For link family A (and A′) and for R2 class (b) we take $s=+3$, $Y'=Y+q/2$; the signs of $\chi$ and $\theta_{YX}$ follow the shear. With $Q_s=(c_u\ c_v)^{-1}$, the adapted frame of a splitting is $g=Q_s^T\mathrm{diag}(6\varsigma_u,2\varsigma_v)Q_s$, $\varsigma_{u,v}=\pm1$.

### 1. Mixing of $U(1)_Y$ and $U_{BL}$ in the readings

Kinetic mixing between abelian factors is generic in the presence of fields charged under both [83–85]; here it also enters through the global form. With the coupling form $\mathcal{T}$ above, whose center block $\tau_z$ is written in the lattice basis $(e_Y,e_{BL})$ with $N_Y=6$, the kinetic mixing $\chi$ and the angle $\theta_{YX}$ enter only $\mathcal{T}$, so the global forms and their self-duality hold for every $|\chi|<1$ and every $\theta_{YX}$. S acts

as $\mathcal{T} \mapsto -g^T \mathcal{T}^{-1} g$ for a frame $g$; in the literal frames of the bidoublet and of link family B, $g = \mathrm{diag}(6s_Y, 2s_B)$ on the center block, and at $\theta = 0$

$$\begin{gathered} \chi' = -s_Y s_B\, \chi, \\ g_Y g'_Y = 12\pi\sqrt{1-\chi^2}, \qquad e_{BL} e'_{BL} = \pi\sqrt{1-\chi^2}. \end{gathered} \tag{A6}$$

On a splitting $u = (1, s)$, $v = (0, 1)$, the single-coupling condition reads

$$\begin{gathered} e_{BL} = \frac{g_I}{2}, \qquad \theta_{BL} = 2\theta_I, \qquad \chi = -\frac{s\, g_Y}{3 g_I}, \\ \theta_{YX} = -\frac{s}{3}\theta_{BL}, \qquad \hat{\tau}'_Y = \hat{\tau}_Y - \frac{s^2}{3}\tau_I, \end{gathered} \tag{A7}$$

with $s \equiv 0 \pmod 6$ for the bidoublet and link family B and $s \equiv 3 \pmod 6$ for R2 class (b) and link family A. The single-coupling requirement can be read in three ways. (a) On the literal splitting only the bidoublet and family B qualify, with $\chi = \theta_{YX} = 0$; R2 and family A admit no single-coupling self-dual theory. (b) On any splitting all four forms qualify, on infinitely many splittings, and every locus has $|\chi| < 1$; the smallest $|\chi|$ is on the literal splitting for the bidoublet and family B, and on $(1, \pm 3), (0, 1)$ for R2 and family A, where $\chi^2 = g_{Y'}^2/(g_{Y'}^2 + g_I^2)$. (c) Shearing only the $G_{\mathrm{SM}}$ circle gives Eq. (A7); at $s = +3$, $\chi = -g_Y/g_I$, $\theta_{YX} = -2\theta_I$ and $1/g_Y^2 = 1/g_{Y'}^2 + 1/g_I^2$. On $\mathrm{Fix}(\mathcal{M})$ with both sectors on the same $(1, s)$ locus, $\chi_{(1)}$ and $\chi_{(2)}$ have the same sign, and at the self-dual point $\chi^2 = s^2/(s^2+3)$: $\chi = \mp\sqrt{3}/2$ for R2 at $s = \pm 3$, and $\chi^2 = \frac{12}{13}$ for the bidoublet at $s = \pm 6$. $\mathcal{M}_P$ gives the same couplings with opposite angles. On spin manifolds the pure period of $\theta_{YX}$ is $8\pi$ and that of $\theta_Y$ is $4\pi$ for the bidoublet at $(1, -6)$ and R2 at $(1, 2)$, the latter consistent with the period $2\pi \cdot 6^2$ of the hypercharge angle for the $\mathbb{Z}_6$ quotient [86], which refers to the angle $\tilde{\theta} = 18\theta_Y$ of the hypercharge field normalized to integer charges $6Y$ ($72\pi/18 = 4\pi$).

In the four link-field classes the lattice basis is again $N_Y = 6$. Family A on $(1, \pm 3), (0, 1)$ has $e_{BL} = g_I/2$, $\chi = \mp g_Y/g_I$ and $\theta_{YX} = \mp 2\theta_I$; A′ the same with $SU_L(2) \leftrightarrow SU_I(2)$; B is unmixed only on the literal splitting; and B′, which has no splitting of the form $(1, s), (0, 1)$, has on $u = (1, 0)$, $v = (c, 1)$ with $c$ odd

$$\begin{gathered} \frac{1}{e_{BL}^2} = \frac{4}{g_I^2} + \frac{36c^2}{g_Y^2}, \\ \chi = -\frac{6c\, e_{BL}}{g_Y}, \qquad \theta_{YX} = -6c\,\theta_Y. \end{gathered} \tag{A8}$$

In A, A′ and B′ no single-coupling locus contains an unmixed point, so self-duality with a single coupling requires $\chi \neq 0$. In B′ the self-dual point has $\chi^2 = 3c^2/(1+3c^2)$ in both sectors. In all four classes the pure periods are $8\pi$ for $\theta_{YX}$, $4\pi$ for $\theta_Y$ and $8\pi$ for $\theta_{BL}$; the full shift lattices of A, A′ and B′ differ from that of B.

## Appendix B: Anomaly coefficients and searches for additions

### 1. Coefficients

We list the triangle-anomaly coefficients of one sector in the left-handed Weyl basis, with $T(\mathrm{fund}) = \frac{1}{2}$ and $BL$ the integer $U_{BL}$ charge $q$. They are group-theory factors of the triangle diagrams; no loop is evaluated. The Witten anomaly of an $SU(2)$ factor is the parity of its number of Weyl doublets [21], which is the test applied here; refined global anomalies [87] are not examined. In all three readings the mirror sector is the parity image, its coefficients are minus the ordinary ones at its own charges, and, with the shared neutrinos, every mixed ordinary–mirror coefficient vanishes identically, since no field carries charges of both sectors.

*Bidoublet.* Besides the charge-independent coefficients of Sec. VII B,

$$\begin{gathered} A_{SU(3)^2 BL} = 2q_Q, \quad A_{SU_I(2)^2 BL} = 3q_Q, \\ A_{SU_L(2)^2 BL} = 3q_Q + \tfrac{q_L}{2}, \quad A_{\mathrm{grav}^2 BL} = 12q_Q + 2q_L, \\ A_{Y^2 BL} = \tfrac{q_Q}{3} + \tfrac{q_L}{2}, \quad A_{Y\,BL^2} = 2q_Q^2 - q_L^2, \\ A_{BL^3} = 12q_Q^3 + 2q_L^3, \end{gathered} \tag{B1}$$

with seven $SU_L(2)$ and six $SU_I(2)$ doublets. $A_{Y^2 BL}$ vanishes at $(3, -2)$, but then $A_{\mathrm{grav}^2 BL} \neq 0$, and $A_{Y\,BL^2}$ and $A_{BL^3}$ have no rational zero. In the adapted basis of class (c), with $q' = q + c\,y$ and $y = 6Y$, every linear coefficient shifts by the hypercharge one, $A_{Xq'} = A_{Xq} + 6c\,A_{XY}$, and in class (c) the most coefficients that vanish together, in the literal basis, are 2 of 13, at $q_Q = 0$.

*R2.*

$$\begin{gathered} A_{SU(3)^2 BL} = q_Q, \quad A_{SU_L(2)^2 BL} = \tfrac{q_L}{2}, \\ A_{SU_I(2)^2 BL} = \tfrac{3q_Q}{2}, \quad A_{\mathrm{grav}^2 BL} = 6q_Q + 2q_L, \\ A_{Y^2 BL} = \tfrac{q_Q}{6} + \tfrac{q_L}{2}, \quad A_{Y\,BL^2} = q_Q^2 - q_L^2, \\ A_{BL^3} = 6q_Q^3 + 2q_L^3, \end{gathered} \tag{B2}$$

in addition to $A_{SU_L(2)^2 Y} = -\frac{1}{4}$, $A_{SU_I(2)^2 Y} = +\frac{1}{4}$ and the Witten counts 1 and 3. Over primitive charges with $0 \le q_Q \le 12$, $|q_L| \le 24$ at least 7 of the 13 coefficients are nonzero, exactly 7 only at $(0, 1)$. In the two-sector theory with shared neutrinos the 24 mixed coefficients ($G_a^2 U'$, $G_a'^2 U$ and the mixed cubic $U(1)$ monomials) vanish identically, and the Witten counts of $(SU_L(2), SU_I(2), SU_L(2)', SU_I(2)')$ are $(1, 3, 1, 3)$.

*Link field.*

$$\begin{gathered} A_{SU(3)^2 BL} = q_Q, \quad A_{SU_L(2)^2 BL} = \tfrac{3q_Q + q_L}{2}, \\ A_{\mathrm{grav}^2 BL} = 6q_Q + 2q_L, \quad A_{Y^2 BL} = \tfrac{q_Q}{6} + \tfrac{q_L}{2}, \\ A_{Y\,BL^2} = q_Q^2 - q_L^2, \quad A_{BL^3} = 6q_Q^3 + 2q_L^3, \end{gathered} \tag{B3}$$

and all other coefficients vanish; $SU_L(2)$ has four doublets and $SU_I(2)$ none. In the sheared basis of family A the coefficients mix, e.g. $A_{SU^2 Y'} = A_{SU^2 Y} + \frac{s}{6} A_{SU^2 BL}$, but whether they cancel does not depend on the basis.

### 2. Search setup

Additions are left-handed Weyl multiplets $(\mathbf{c}, \mathbf{d}_L, \mathbf{d}_I)_{Y,q}$ with $\mathbf{c} \in \{\mathbf{1}, \mathbf{3}, \bar{\mathbf{3}}\}$, $\mathbf{d}_L, \mathbf{d}_I \in \{\mathbf{1}, \mathbf{2}\}$ and multiplicity at most six (the total singlet excluded). The conditions are that all 13 perturbative coefficients vanish and that both doublet counts are even. A set keeps a global form if every added multiplet is trivial on the subgroup by which that form is a quotient; the variant "any" drops this condition and breaks the gauge group. The search boxes are

N: $|Y| \le \frac{3}{2}$, $|q| \le 12$;

W: $|Y| \le \frac{5}{2}$, $|q| \le 20$;

R: $|Y| \le \frac{3}{2}$, $|q| \le 9$ (R2, one and two sectors).

The minima are optima of integer linear programs. Unless stated otherwise they rest on floating-point branch-and-bound certificates of the solvers, every reported set is verified in exact arithmetic, and smaller sets outside the box are not excluded.

### 3. Minimal sets

The minimal numbers of added components are collected in Table V; we add what the table does not show (the explicit supports are not shown). For the bidoublet, at the 30 points $q_Q \in \{1, 3, 5\}$, $q_L$ even with $2 \le |q_L| \le 10$, the minimal compatible addition has 15 components in box N only at $(1, -6)$, where its support is unique, and in box W at $(1, -10)$, $(1, -6)$, $(1, 4)$ and $(1, 6)$; the other points need 16 to 21. For class (c) the minimum does not depend on $q_L$, and at $q_Q = 0$ it is 13 in both boxes, with a unique minimal support that keeps the class-(c) form faithful. For R2, in box R with the self-dual form kept, the cheapest anomaly-free sector costs 4 components at $(q_Q, q_L) = (0, 1)$, with a colorless $SU_I(2)$ doublet, $(\mathbf{1}, \mathbf{1}, \mathbf{2})_{-1/2,0} + (\mathbf{1}, \mathbf{2}, \mathbf{1})_{1/2,-1}$; 12 at $(1, 0)$ if $SU_I(2)$ acts on quarks only, namely an ordinary quark doublet $(\mathbf{3}, \mathbf{2}, \mathbf{1})_{1/6,0}$ and the conjugate of $Q$, which make $Q \oplus \bar{Q}$ vector-like under $SU_I(2) \times U_{BL}$; and 15, a vector-like completion, otherwise. Class (a), including the $B - L$ point $(1, -3)$, has no self-dual form whatever is added. For the link field, at the 36 points $q_Q \in \{2, 4, 6\}$, $q_L$ even with $0 < |q_L| \le 12$, families A and B and $SU_I(2)$-singlet additions have the same optimum in box N at every point; the smallest is 9 at $(2, -6)$, proven exactly, with four minimal supports, and every other point needs at least 10. In box W the optimum is 9 at $(2, -6)$ and at $(4, -12)$; at the other points only upper bounds are established.

## Appendix C: Descent through the link-field breaking

This appendix gives the criterion and the theorem behind the statement of Sec. VII C: whether sector 2 $= S_f$(sector 1) on $\mathrm{Fix}(\mathcal{M})$, for a frame $f$, descends to the unbroken group $H_0$ of the link field when $Y$–$BL$ mixing is included. The analysis is at tree level and classical, with the infrared coupling the restriction of the ultraviolet coupling form to the unbroken torus and $|\chi| < 1$; the conventions are those of App. A. After $\langle V \rangle = v\, 1_2$, the identity component $H_0$ of the unbroken group is isomorphic to $G_{\mathrm{SM}}$ in classes A and A′ at every scanned charge point, and in classes B and B′ exactly when $3 \nmid q_V / \gcd(q_Q, q_L, q_V)$; it is Langlands self-dual exactly there. With $n$ the primitive cocharacter of the center circle of $H_0$, its couplings are $\tau_X = n^T \tau_z n$ and $\tau_D = \tau_L + \tau_I$, and the $S$ of $H_0$ acts as $\tau_X \to -36/\tau_X$. With $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, define

$$\begin{gathered} w = f^{-1} n, \qquad m = J^T w, \\ \Pi_{nn} = n^T \tau_z n, \quad \Pi_{mm} = m^T \tau_z m, \quad \Pi_{nm} = n^T \tau_z m, \\ \Delta_f = (n^T f^{-1} n)^2, \qquad \kappa_f = 1 - \Delta_f/36. \end{gathered} \tag{C1}$$

Then

$$\tau_X^{(1)} \tau_X^{(2)} + 36 = \frac{36\,(\kappa_f \Pi_{nn} \Pi_{mm} - \Pi_{nm}^2)}{\Pi_{nn} \Pi_{mm} - \Pi_{nm}^2}, \tag{C2}$$

so the $U(1)$ relation $\tau_X^{(2)} = -36/\tau_X^{(1)}$ holds if and only if $\Pi_{nm}^2 = \kappa_f\, \Pi_{nn} \Pi_{mm}$, and the $SU(2)_D$ relation $\tau_D^{(2)} = -1/\tau_D^{(1)}$ holds if and only if $\tau_L/\tau_I = e^{\pm 2\pi i/3}$.

The sign of $\kappa_f$ is fixed in every frame. For the link-field group at any charge point where $H_0 \cong G_{\mathrm{SM}}$, and for every Langlands self-duality $F : X_* \to X^*$ with frame $f$,

$$\begin{gathered} n^T f^{-1} n \equiv \pm 12 \pmod{36}, \\ \Delta_f \in 144\,\{j^2 : 3 \nmid j\}, \qquad \kappa_f \le -3, \end{gathered} \tag{C3}$$

and every residue $\pm 12$ modulo 36 is attained, so $\min \Delta_f = 144$ and $\Delta_f = 36$, for which $\kappa_f = 0$, never occurs. The proof uses the glue element $\gamma = n/6 + \frac{1}{2}(\alpha_L^\vee + \alpha_I^\vee) + \omega^\vee$ that $H_0 \cong G_{\mathrm{SM}}$ provides in the cocharacter lattice, with $\omega^\vee$ a fundamental coweight of $SU(3)$: the pairing $\langle F\gamma, \gamma \rangle$ must be an integer, and modulo 1 the two $SU(2)$'s contribute $\pm\frac{1}{2} \pm \frac{1}{2} \equiv 0$ and $SU(3)$ contributes $\pm\frac{2}{3}$. Since $\kappa_f < 0$, the relation does not descend on any single-coupling locus, for any $\theta$ (on all 236 combinations of class, pairing, splitting and adapted sign that we examined), nor anywhere at $\theta = 0$, where the $\Pi$'s are imaginary. Off these loci with $\theta \neq 0$ it descends on a nonempty set in every class and frame, with no constraint on $(g_Y, e_{BL}, \chi)$, but the angles never all vanish and at weak coupling amount to hundreds to thousands of periods.

Descent may also be read as holding up to the $\theta$ periods of the infrared. On spin manifolds the ultraviolet shifts that are symmetries restrict onto the infrared period lattice, and shifts never relax the size of the angles on which $S_f$ acts. If the mirror map may be $S_f T_t$, with $T_t$ the shift $\mathcal{T} \to \mathcal{T} + t$ of the coupling form of

App. A by an element $t$ of the ultraviolet period lattice, every solution of the strict relation reduces into a fundamental domain, so order-one physical angles suffice for every $(g_Y, e_{BL}, |\chi| < 1)$ and every $g_L/g_I$; the size of $t$ depends on the frame and is not physical. What remains are obstructions that are not about periods: $\Delta_f \geq 144$, the bound $\operatorname{Im}\tau_X^{(1)} \operatorname{Im}\tau_X^{(2)} \leq 36$ that the $U(1)$ relation implies (a lower bound on the product of the two $U(1)$ couplings), and, in classes B and B′, the condition $3 \nmid q_V / \gcd(q_Q, q_L, q_V)$ for $H_0 \cong G_{\rm SM}$.

## Appendix D: Supersymmetry and the mirror models

The mirror models of Ref. [10] are described there in supersymmetric language: SMM4 as a massless gauge supermultiplet (tested in Sec. II C), and SMM4b through a balance of bosonic and fermionic degrees of freedom. We ask whether these structures can supply a setting in which $S$ is exact (App. B of Ref. [13]).

### 1. The pseudo-supersymmetry of Ref. [10]

In a supersymmetry whose supercharges commute with the internal gauge generators, as the Haag–Łopuszański–Sohnius theorem [19] requires for an internal symmetry outside the R-symmetry, the superpartner of a gauge boson is in the adjoint representation, and $\operatorname{Str} g = 0$ for every gauge element $g$. The pseudo-supersymmetry of Ref. [10], which pairs matter fermions with gauge bosons and the Higgs scalars with the right-handed neutrinos, is only the statement $\operatorname{Str} 1 = n_b - n_f = 0$: in SMM4 as given there ($n_b = n_f = 30$) the supertrace of the central element $\omega 1_3$ of $SU(3)$ is 36, and in SMM4b it is 108; no Standard Model Weyl field has $Y = 0$, so none can be a gaugino. Nor does it make the couplings marginal. If the six Higgs fields of Ref. [10] were elementary doublets, the three-generation model would have $b_0 = (7, \frac{7}{3}, -\frac{23}{3})$ for $(\mathfrak{su}_3, \mathfrak{su}_2, \mathfrak{u}_{1,Y})$, with $b_0$ as in Sec. III F, so $S$, $\mathcal{M}$ and $\mathcal{M}_P$ could not be exact symmetries at a fixed coupling; in the reading used here they are quark condensates, and without them the conclusion is unchanged. With $n_W$ Weyl fermions and $n_S$ real scalars per generator in the adjoint, the condition $n_b = n_f$ gives

$$b_0 = (4 - n_W)\, C_2(G), \tag{D1}$$

so pseudo-supersymmetry together with one-loop conformality holds only for $(n_W, n_S) = (4, 6)$, the $\mathcal{N} = 4$ content. The heterotic Calabi–Yau compactification of Ref. [10] has $\mathcal{N} = 1$ per sector, and exact $S$ would need $\mathcal{N} = 4$, which only a torus supplies.

### 2. Supersymmetry with gauge-charged supercharges

A different way to make the pairings of Ref. [10] supersymmetric is to let the supercharges carry gauge charges, the loophole that the Coleman–Mandula [18] and Haag–Łopuszański–Sohnius [19] theorems leave open. We restrict to global supersymmetry, and we count supersymmetry parameters dressed with Wilson lines as global. The minimal representation of the supercharges that pairs every species of a generation with a gauge boson is one generation itself, of complex dimension 15. In four dimensions the super-Jacobi identities admit such an algebra, but its massless multiplets then contain helicities $|h| \geq 4$, which interacting massless particles without gravity exclude [88–90], and supercharges of dimension at most 4 pair at most three species with a gauge boson, and $Q_L$ never. Gauging is the decisive obstruction. The commutator of a gauge transformation $\alpha(x)$ and a supersymmetry transformation is a supersymmetry transformation with parameter $-iq\alpha(x)\epsilon$. With the gauge field outside any multiplet and a charged chiral multiplet, invariance forces $\epsilon = 0$ at a point with generic $F_{\mu\nu}$, and at $F = 0$ it requires exactly $D_\mu \epsilon = 0$, whose integrability condition $[D_\mu, D_\nu]\epsilon = -iqF_{\mu\nu}\epsilon$ must hold in all four directions; with a charged abelian vector multiplet, the three-gaugino terms force $\epsilon = 0$ even at $F = 0$. Within these ansätze, which are classical, abelian and linear in the fields, no global supersymmetry, with constant or Wilson-line-dressed parameters, has a supercharge charged under a dynamical four-dimensional gauge field. In two dimensions, where only $D_- \epsilon_+ = 0$ is needed, such an algebra exists and Wilson-line-dressed parameters restore it under gauging; that setting is the subject of App. C of Ref. [13].

## Appendix E: An $\mathcal{N} = 2$ model with gauge group $G_{\rm SM}$

The Seiberg–Witten tool of Sec. III F completes each chiral superfield of SMM4 into a hypermultiplet in order to use the exact duality of $\mathcal{N} = 2$ $SU(2)$ with four flavors [29]. Here we take this completion seriously as a model. A hypermultiplet in a representation $R$ contains a left-handed Weyl fermion in $R$ and one in $\bar{R}$; the second, written as a right-handed field, is in $R$. If the first is a Standard Model fermion, the second is its mirror fermion, so an $\mathcal{N} = 2$ theory with gauge group $G_{\rm SM}$ contains the ordinary and the mirror fermions of one sector as superpartners. $\mathcal{N} = 2$ gauge theories and their Higgs mechanism go back to Ref. [91], and Ref. [92] studies $\mathcal{N} = 2$ models of this kind. We first describe the model on a single $G_{\rm SM}$ and its duality structure, and then its unfolding into two sectors, $G_{\rm SM} \times G'_{\rm SM}$. All statements are classical or one loop, except where the exact results of $\mathcal{N} = 2$ theories are quoted from the literature.

### 1. One $G_{\rm SM}$

Each Standard Model chiral field $Q, u^c, d^c, L, e^c$, and optionally $\nu^c = (\mathbf{1},\mathbf{1})_0$, gets one hypermultiplet per generation, and $n_H$ Higgs hypermultiplets $(\mathbf{1},\mathbf{2})_{1/2}$ are added. Per generation this gives 16 Standard Model and 16 mirror Weyl components (15 and 15 without $\nu^c$); each Higgs hypermultiplet adds four higgsino components, and the vector multiplets add 24 gaugino components. For $n_g \in \{1,3\}$ and $n_H \in \{0,1,2\}$ every gauge anomaly vanishes and the $SU(2)$ Witten count is even. The kernel of $SU(3)\times SU(2)\times U(1)_Y$ on the content is $\mathbb{Z}_6$, so all four global forms, $G_{\rm SM}$ included, are admitted. The $\nu^c$ hypermultiplet is a free singlet. The Standard Model Yukawa couplings are products of three hypermultiplet chiral fields, not of the $\mathcal{N}=2$ form $\tilde{q}\Phi q$, so they are absent, and the Higgs doublets cannot sit in a vector multiplet of a single $G_{\rm SM}$. The hypermultiplet partners are exactly the mirror fermions of SMM4, $Q'_R, u'_L, d'_L, L'_R, e'_L, \nu'_L$, with $G'_{\rm SM}$ identified with $G_{\rm SM}$; the two-sector SMM4 fermions of one generation are the one-generation hypermultiplet fermions, and the four Higgs doublets $\phi_u, \phi_d, \phi'_u, \phi'_d$ are the scalars of $n_H = 2$ Higgs hypermultiplets. Compared with SMM4, $\mathcal{N}=2$ adds 32 complex sfermions per generation, 24 gaugino components and 12 adjoint complex scalars, and it has one $G_{\rm SM}$ instead of two.

With $\beta(g) = -b\,g^3/16\pi^2$ as in Sec. III F, the one-loop coefficients for $(SU(3), SU(2), U(1)_Y)$ are, as in Paper I (its App. B),

$$b = \Big(6 - 4n_g,\ \ 4 - 4n_g - n_H,\ \ -\tfrac{20}{3}n_g - n_H\Big). \qquad \text{(E1)}$$

For one generation $SU(3)$ is asymptotically free, $SU(2)$ is conformal at $n_H = 0$ and infrared free otherwise; for three generations every factor is infrared free, and with the $\mathcal{N}=2$ content from $M_Z$ on, the Landau poles lie below $M_P$ (that of $SU(3)$ at $6.5\times10^5$ GeV). If the Standard Model runs up to a common threshold $M_2$ above which the $\mathcal{N}=2$ content enters, all poles lie above $M_P$ only for $M_2 \gtrsim 1.2\times10^8$, $2.5\times10^8$ and $1.3\times10^9$ GeV for $n_H = 0, 1, 2$.

At the conformal point $(n_g, n_H) = (1, 0)$ the $SU(2)$ has $N_f = 4$ doublet hypermultiplets, three from $Q$ (one per color) and one from $L$. As a set of $G_{\rm SM}$ chiral fields this is the conformal completion of Sec. III F without the $\phi$ hypermultiplet, so the duality group $\Gamma^0(2)$, the strong–weak element $TST^{-1}$ at $\theta_2 = \pi$, $g_2^2 = 8\pi$, and the mirror elements built from it apply unchanged. The flavor masses set by the spectator Coulomb parameters are $a_1 + a_Y/6$, $a_2 + a_Y/6$, $-a_1 - a_2 + a_Y/6$ and $-a_Y/2$; their sum vanishes identically, by the same sum that makes $A_{SU(2)^2Y} = 0$. The $SU(3)\times SU(2)$ part is the linear quiver [2]–$SU(3)$–$SU(2)$–[1] with $(\mathbf{3},\mathbf{2})$ as the bifundamental: the $SU(2)$ node is balanced, and the $SU(3)$ node, with $N_f = 4 < 6$, is asymptotically free. The measured coupling gives $\tau_{SW} = 2\tau_2 = 59.16\,i$ at $\theta_2 = 0$, so the self-dual point $\alpha_2 = 2$ lies a factor 59.16 above it; treating $SU(3)$ and $U(1)_Y$ as spectators is exact only for $g_3, g_Y \to 0$, which $SU(3)$ approaches in the ultraviolet.

### 2. Curve and duality group

In the M-theory form of Ref. [93] the curve of the quiver [2]–$SU(3)$–$SU(2)$–[1] has genus 3, the rank of $SU(3)\times SU(2)$, and the relative bending of its branches, $(2, 0)$, equals the coefficients $b$ of $SU(3)$ and $SU(2)$. $U(1)_Y$ is infrared free and enters only through the brane positions, which reproduce the flavor masses above (with the orientation of $L$ of Eq. (2.9) of Ref. [93]); the $\mathbb{Z}_6$ of $G_{\rm SM}$ acts trivially on every hypermultiplet and on the curve, so the global form only selects the line lattice [94]. In the labeling $t_1 = \lambda(\tau_{SW})$ of the cross-ratio parameter of the curve of Ref. [95], in which weak coupling is $t_1 \to 0$, the duality group $\Gamma^0(2)$ acts as $t_1 \to 1/t_1$, and the fixed point $\tau_{SW} = 1 + i$ of $TST^{-1}$ sits at $t_1 = -1$, the $\mathbb{Z}_2$ orbifold point of Ref. [93]. The Langlands map of Paper I is not a duality of this family, for a reason independent of the one in Sec. III F: in $\tau_2$ the duality group is $\Gamma_0(2)$, which $S_\Phi : \tau_2 \to -1/\tau_2$ conjugates to $\Gamma^0(2) \neq \Gamma_0(2)$. Of the 1152 automorphisms of the $D_4$ flavor algebra, 24 preserve the gauged $\mathfrak{su}(3)\oplus\mathfrak{u}(1)_Y$; all fix the bad spinor class of Sec. III F, and 12 exchange the vector class with the other spinor class, as $TST^{-1}$ does. That $\Gamma^0(2)$ is then a duality of the whole quiver follows the logic of gauging a flavor group preserved by the duality [95, 96]; it is an argument, not a computation. On one sector, the stabilizer of $\tau_{SW} = 1 + i$ in $\Gamma^0(2)$ extended by parity contains $\tau_{SW} \to \bar{\tau}_{SW}/(\bar{\tau}_{SW} - 1) = T^2PTST^{-1}$, parity composed with the strong–weak element: the single-$G_{\rm SM}$ analog of $\mathcal{M}_P$.

### 3. Parity

The $\mathcal{N}=2$ Lagrangian on one $G_{\rm SM}$, with its gauge Yukawa couplings, superpotential $W = \sqrt{2}\tilde{q}\Phi q + m\tilde{q}q$ and potential, has a parity $P_{\mathcal{N}=2}$ that maps $x \to (t, -\mathbf{x})$, each Standard Model Weyl fermion to its hypermultiplet partner ($Q_L \to Q'_R$, $u_R \to u'_L$, $d_R \to d'_L$, $L_L \to L'_R$, $e_R \to e'_L$, $\nu_R \to \nu'_L$), the gaugino to the conjugate of the adjoint chiral fermion, and $\phi \to \phi^*$, with the hypermultiplet scalars fixed up to phases. It normalizes $SU(2)_R$ and inverts $U(1)_R$. It needs the hypermultiplet masses to be real up to a common phase (in the frame $\phi \to \phi^*$), and $\theta_i \in \{0, \pi\}$, since $F\tilde{F}$ is odd. It carries the matter action of $\mathcal{M}_P$, while the $S$ part is absent: it is what remains of $\mathcal{M}_P$ on a single $G_{\rm SM}$. It exchanges the gaugino with the adjoint chiral fermion, so it maps one $\mathcal{N}=1$ subalgebra to another, but it is a symmetry of the full $\mathcal{N}=2$ Lagrangian, superpotential included. Without spacetime parity and without a gauge outer automorphism, no gauge-covariant map exchanges all Standard Model fermions with their partners, because $Q, u^c, d^c, L, e^c$ are in complex representations (only the $\nu^c$ pair, in a real representation, can be

exchanged alone). $P_{\mathcal{N}=2}$ differs from the internal mirror parity of Ref. [92] [their Eq. (2.7)]. These statements are classical; the $U(1)_R$ anomaly is not included.

#### 4. Two sectors: the quiver $G_{\rm SM} \times G'_{\rm SM}$

The mirror fermions are hypermultiplet partners only when the two sectors share one gauge group: a hypermultiplet of $G_{\rm SM} \times G'_{\rm SM}$ in $\rho$ holds fermions in $\rho$ and $\bar\rho$, while SMM4 pairs $\psi \in (R, \mathbf{1})$ with $\psi' \in (\mathbf{1}, \bar R)$, which has this form only for $\nu^c$. A two-sector $\mathcal{N} = 2$ version must therefore reach the single $G_{\rm SM}$ by breaking. We unfold the model into a quiver $G_{\rm SM} \times G'_{\rm SM}$ with a link of two bifundamental hypermultiplets,

$$B_3 \in (\mathbf{3},\mathbf{1})_{-1/3} \otimes (\bar{\mathbf{3}},\mathbf{1})'_{1/3}, \qquad B_2 \in (\mathbf{1},\mathbf{2})_{1/2} \otimes (\mathbf{1},\mathbf{2})'_{-1/2}, \tag{E2}$$

the block-diagonal part of $\mathbf{5} \otimes \bar{\mathbf{5}}'$ with $\mathbf{5} = (\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$. The vacuum $q = v1_5$, $\tilde q = w1_5$ is D- and F-flat for all $v, w$, because $Y$ is traceless on $\mathbb{C}^5$ ($B_3$ alone has no flat direction). It breaks the quiver to its diagonal, with $1/g^2_{\rm diag} = 1/g^2 + 1/g'^2$ for each factor; at $w = 0$ the squared vector masses are $(g^2 + g'^2)|v|^2$ for $SU(3)$ and $SU(2)$ and $\frac{5}{3}(g_Y^2 + g_Y'^2)|v|^2$ for $U(1)$. Twelve long massive vector multiplets (massive $\mathcal{N} = 2$ multiplets without central charge [20]) eat the adjoint part of the link, and one neutral hypermultiplet $(\mathbf{1},\mathbf{1})_0$ is left, a modulus of the Higgs branch. Each hypermultiplet of the single-$G_{\rm SM}$ model is placed on one node or split between them ($Q$ as $(\mathbf{3},\mathbf{1})_y \otimes (\mathbf{1},\mathbf{2})'_{1/6-y}$, $e^c$ as $(\mathbf{1},\mathbf{1})_{1/2} \otimes (\mathbf{1},\mathbf{1})'_{1/2}$); all contents built this way (864 for $n_g = 1$ and 215040 for $n_g = 3$, $n_H \le 2$, counting the placements of each field as a multiset) reduce on the diagonal to the single-$G_{\rm SM}$ model, the eaten adjoint and the modulus.

No such content is symmetric under the exchange of the two nodes when $n_g$ is odd. A swap-invariant hypermultiplet $(R_1, R_2)$ has $\dim R_1 = \dim R_2$, and among the single-$G_{\rm SM}$ fields only the split $e^c$ and $\nu^c$ arise in this way; in a swap-invariant content $Q, u^c, d^c, L$ and the Higgs fields therefore occur an even number of times on the diagonal, and none of the enumerated contents is swap invariant. With $k$ generations and one Higgs hypermultiplet on each node the diagonal has $n_g = 2k$ and $n_H = 2$. So a swap-symmetric $\mathcal{N} = 2$ quiver $G_{\rm SM} \times G'_{\rm SM}$ and an odd number of generations exclude each other, and the model we use is an asymmetric quiver, which gives the single-$G_{\rm SM}$ model exactly for $n_g = 1$ and 3. Without neutrino sharing the 32 Weyl fields of SMM4 reduce to the model with $\nu^c$, whose hypermultiplet has the quantum numbers of the link modulus; with the sharing of Sec. III B the 30 distinct Weyl fields reduce to the model without $\nu^c$, in which $\nu_L$ is the neutral component of $L$ and $\nu_R$ that of its partner, the mirror doublet $L'_R$.

The content is vector-like; all 16 triangle anomaly coefficients vanish and both Witten counts are even. When every hypermultiplet is a representation of $G_{\rm SM} \times G'_{\rm SM}$ (all placements except the split $e^c$), $G_{\rm SM} \times G'_{\rm SM}$ acts faithfully, and of its gluings only $G_{\rm SM} \times G'_{\rm SM}$ itself gives exactly $G_{\rm SM}$ on the diagonal after the breaking. At one loop the link alone contributes $b = (3, 2)$ to the non-abelian factors of each node, and for one generation on the first node the node coefficients are $(-1, -2\,|\,3, 2)$ for $n_H = 0$; in every content the sum over the two nodes equals the diagonal $b$ of Eq. (E1). No content has both $SU(2)$ nodes conformal: the conformal $SU(2)$ of the single-$G_{\rm SM}$ model exists only on the diagonal. Since a massive long $\mathcal{N} = 2$ vector multiplet does not contribute to $b$, $\tau_{\rm diag} = \tau + \tau'$ at every scale at one loop. On the fixed locus of $\mathcal{M}_P$ in frame $D$, $\tau_{\rm diag} = 8\pi i/g^2$; the locus is RG-stable if and only if the two nodes have equal $b$, as for swap-symmetric contents; for one generation on one node it drifts.

The mirror elements act on the quiver as the node exchange with $C$ on none, one or both nodes. The complex link is preserved only by $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$, which reinstates the compatibility rule of Paper I. A content with one generation on the first node is preserved by none of the four elements: $\mathcal{M}_P$ maps it to its swap image, a different quiver with the same diagonal theory, while a content with one generation and one Higgs hypermultiplet on each node is preserved by $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$. For odd $n_g$ the mirror elements are therefore not symmetries of one quiver but dualities between quivers that share their diagonal theory. Both $\mathcal{M}_P$ and $\tilde{\mathcal{M}}$ map the diagonal subgroup to itself, and descend to parity ($\tau_{\rm diag} \to -\bar\tau_{\rm diag}$) and to the identity on the diagonal. On the link modulus, with the standard representatives of the field maps, $\mathcal{M}_P$ fixes the vacuum (after a flavor rotation) if and only if $vw$ is real, and $\tilde{\mathcal{M}}$ if and only if $|v| = |w|$; these loci depend on the representative chosen, while $\mathcal{M}_P \circ (C \otimes C)$ acts as an $SU(2)_R$ rotation by $\pi$ and $\tilde{\mathcal{M}} \circ (C \otimes C)$ fixes every vacuum. The lift of Sec. VI D survives in its option (ii): the central element $z$ fixes the link, so $\mathcal{X}_{\rm ii} = \mathcal{M}_P \circ z$ is $\mathcal{M}_P$ up to a gauge transformation on the quiver and on the diagonal. Option (i) does not: with the link, the only fermion-sign symmetries of the $\mathcal{N} = 2$ Yukawa couplings are 1, $(-1)^F$ and signs on free singlet hypermultiplets, and $(-1)^{F'}$ is not among them. On the diagonal, $P$ exchanges $\nu_L \in L$ with $\nu_R$ in the same hypermultiplet, as the mirror map of Sec. III B does.

## Appendix F: Models, samples and vacua of the SMM4b chains and of the two-doublet model

This appendix collects the models, samples and numbers behind Secs. V B, V E and VI E. All SMM4b statements are at leading order in $1/N_c$, in the gaugeless mean field of the chains, on the $U(1)_{\rm em}$-aligned branch; minima are found by multi-start searches, so counts of minima are lower bounds, and all frequencies refer to the samples of App. F 1.

### 1. The mean-field model and the samples

*Mean-field model of the chains.* Per sector the quark condensate is a complex $6 \times 6$ matrix $\mathcal{Q}_{(ia),(jt)} \sim q_{L,ia}\bar{q}_{R,jt}$, with generation indices $i, j$, the isospin index $a$ of the doublet and the right-handed type $t = u, d$: 72 real components, normalized so that its singular values $s_k$ are the dynamical quark masses. With a Euclidean four-momentum cutoff $\Lambda = 1$ and $N_c = 3$ the leading-$N_c$ potential is

$$V(\mathcal{Q}) = \mathrm{vec}(\mathcal{Q})^\dagger K\,\mathrm{vec}(\mathcal{Q}) - \frac{2N_c}{16\pi^2}\sum_{k=1}^{6} F(s_k^2), \qquad \text{(F1)}$$

where $F(x) = \int_0^1 dt\, t\ln(t + x)$, the second term is the quark loop, and for one channel the stationarity condition is Eq. (11). The form $K$ is block diagonal, $K_{(ia,jt),(i'a',j't')} = \delta_{aa'}\delta_{tt'}(K_t)_{(ij),(i'j')}$, with $K_t = G_t^{-1}$ and $G_t$ the positive hermitian coupling form of the scalar channel $\bar{t}_R q_L$ on $\mathbb{C}^3_{\mathrm{gen,L}} \otimes \mathbb{C}^3_{\mathrm{gen,R}}$. With $S_3$ acting as $\Omega \otimes \Omega$, $\Omega$ a $3 \times 3$ permutation matrix of the generations, the invariant forms span a real space of dimension 14, so chain N2 has 14 real parameters per type and 28 per sector: hypercharge forbids an up–down term and $\mathbb{Z}_3^A$ the holomorphic $O_{quqd}$ term. In the near-$G_{54}$ chain, Eq. (13), $D_t$ lies in the 13-dimensional traceless invariant subspace. All surveys use the aligned embedding $\mathcal{Q}_{(iu),(ju)} = (m_u)_{ij}$, $\mathcal{Q}_{(id),(jd)} = (m_d)_{ij}$, in which $V = V_u(m_u) + V_d(m_d)$, and test stability in all 72 directions. A *quark type* is the up or the down channel of one coupling set, two per vacuum.

The vacuum values follow from the left-handed vacuum polarization at zero momentum,

$$\begin{aligned} v_Z^2 &= \frac{N_c}{16\pi^2}\sum_f I(m_f^2, m_f^2), \\ v_W^2 &= \frac{N_c}{8\pi^2}\sum_{i,j} |V_{ij}|^2 I(m_{u_i}^2, m_{d_j}^2), \end{aligned} \qquad \text{(F2)}$$

with $V$ the Cabibbo–Kobayashi–Maskawa (CKM) matrix of the condensate; $v_Z$ sets the $Z$ mass and $v_W$ the $W$ mass. We use three loop functions $I(a, b)$ (the three schemes of the vacuum-value loop): $\overline{\mathrm{MS}}$ at $\mu = \Lambda$, $I_D = \frac{a+b}{2} - \frac{a^2 \ln a - b^2 \ln b}{a-b}$, which is primary; a hard cutoff with the mass-independent quadratic divergence subtracted, $I_C = \int_0^1 dx\, \frac{(a+b)x+ab}{(x+a)(x+b)}$; and a Pagels–Stokar-type interpolation, $I_{PS} = \int_0^1 dx\, \frac{x(a+b)/2}{(x+a)(x+b)}$. At $a = b$ they behave as $2a\ln(1/a)$, $2a\ln(1/a)$ and $a\ln(1/a)$. With $I_D$, $v_Z^2$ equals Eq. (12), which has the leading logarithm of the Pagels–Stokar relation [35] and defines $v$; $I_{PS}$ has half that leading logarithm, so it does not give the same $v$ and enters only through ratios. The cutoff in GeV is $\Lambda = v/\sqrt{v_Z^2}$ with $v = 246.22$ GeV: 860–1176 GeV on the chain-N2 vacua of the main sample below and 851–1164 GeV on its near-$G_{54}$ vacua. The asymmetries between a mirror configuration (primed) and the ordinary one are

$$\varepsilon_{\mathrm{rms}} = \Big(\frac{\sum_f m_f'^2}{\sum_f m_f^2}\Big)^{1/2} - 1, \quad \varepsilon_Z = \frac{v_Z'}{v_Z} - 1, \quad \varepsilon_W = \frac{v_W'}{v_W} - 1, \qquad \text{(F3)}$$

of which $\varepsilon_{\mathrm{rms}}$ is scheme independent, and energy differences are given as $\Delta E/E_{\mathrm{cond}}$, with $E_{\mathrm{cond}} = [V(0) - V_{u0}] + [V(0) - V_{d0}]$ the condensation energy of the two types. The Jarlskog invariant is $J = \mathrm{Im}(V_{12}V_{23}V_{13}^*V_{22}^*)$ with masses in ascending order, zero for a degenerate same-type pair. The vacuum-value loop contains the trace $\mathrm{tr}(\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5)k_\alpha k_\beta$, which vanishes by antisymmetry.

*Coupling ensembles of the chains.* In chain N2 each $G_t$ ($t = u, d$ independently) is drawn as $G = \frac{1}{6}\sum_{\Omega\in S_3}(\Omega \otimes \Omega)A(\Omega \otimes \Omega)^T$ with $A = BB^\dagger + 0.1\mathbb{1}$, $B$ a $9 \times 9$ matrix of independent complex Gaussian entries (real and imaginary parts standard normal), rescaled so that its largest eigenvalue is $rG_c$ with $r$ uniform in $[1.2, 2.5]$: complex couplings, all channels supercritical. In the near-$G_{54}$ chain $g_u, g_d$ are uniform in $[1.2, 2.5]$ and $D_t$ is the traceless part of the $S_3$ average of $(B + B^\dagger)/2$, divided by its spectral norm, with $\epsilon_S \in \{0.01, 0.05, 0.2\}$. The cutoff is $\Lambda = 1$ in every survey. Minima are found by multi-start L-BFGS-B minimization, with a Newton polish in every sample except the first. The samples, and the numbers of this paper that belong to them, are: (1) the *count sample* (60 coupling pairs, 10 starts per type), with 61, 63 and 65 pNGBs in 10, 29 and 21 vacua; (2) the *curvature sample* (24 pairs), with exactly 4 zero modes in each vacuum (Sec. V B); (3) the *gauge-loop sample* (120 pairs), in which 23 coupling sets have a lower misaligned gaugeless minimum and none is overturned by the gauge loops; (4) the *CP sample* (240 complex and 120 real pairs, 16 starts per type), with 109 vacua of 65 pNGBs, 7 of them with $J \neq 0$, 2 from real couplings (Sec. VI A); (5) the *main sample* (120 chain-N2 vacua and 30 near-$G_{54}$ vacua at each $\epsilon_S$, 12 starts per type), which carries the class counts, curvatures, spreads and quark masses of Secs. V B and V D, the axion masses of Sec. V E and the vacua of Sec. VI E; pNGB classes are taken at a relative mass tolerance $2 \times 10^{-4}$ and checked at $10^{-6}$, and the robust 45 class of chain N2 has 55 vacua (56 at $2 \times 10^{-4}$); (6) the *competitor search* on the couplings of the main sample, with 4000 random starts per quark type at scales 0.25, 0.5, 0.8 and 1.2, seeded with the known minima, followed by up to four rounds of starts along the eight softest Hessian modes, 40 random perturbations of each minimum and descents from saddles; (7) the *scaling set*, 30 fixed pairs $(g_t, D_t)$ at eight values of $\epsilon_S$ from 0.2 to 0.001, of which 5 have a competing minimum at small $\epsilon_S$. Every frequency and range quoted for the chains belongs to the sample named with it.

*Two-doublet model.* With $\Phi_2 = \phi_u$ and $\Phi_1 = i\sigma_2\phi_d^*$,

$n_i = \Phi_i^\dagger \Phi_i$ and $z = \Phi_1^\dagger \Phi_2$, the potential is

$$\begin{aligned} V = {} & m_{11}^2 n_1 + m_{22}^2 n_2 - m_{12}^2 (z + z^*) + \tfrac{\lambda_1}{2} n_1^2 + \tfrac{\lambda_2}{2} n_2^2 \\ & + \lambda_3 n_1 n_2 + \lambda_4 |z|^2 + \tfrac{\lambda_5}{2} (z^2 + z^{*2}) \\ & + (\lambda_6 n_1 + \lambda_7 n_2)(z + z^*), \end{aligned} \tag{F4}$$

with real parameters in units with $|m^2| \leq 1$, each drawn uniformly: $m_{11}^2, m_{22}^2, m_{12}^2 \in [-1, 1]$, $\lambda_{1,2} \in [0.1, 4]$, $\lambda_{3,4,5} \in [-4, 4]$, and $\lambda_{6,7} \in [-1, 1]$ (general) or $\lambda_6 = \lambda_7 = 0$ (softly broken $\mathbb{Z}_2$). The *unbiased survey* has 150 potentials per family, 24 starts each; the *targeted scan* has 15000 draws per family, with minima on the neutral CP-even plane verified in all eight directions. In both, the quartic part is required to have its minimum above $10^{-3}$ on the unit sphere (in the targeted scan, for the candidate pairs), and the targeted scan defines $\varepsilon = v_{\rm other}/v_{\rm deepest} - 1$ with $v^2 = v_u^2 + v_d^2$. The *representative point* is $(m_{11}^2, m_{22}^2, m_{12}^2, \lambda_1, \ldots, \lambda_7) = (-1, -0.8, 0, 1, 1.2, 0.6, -0.5, -0.3, 0, 0)$. The *QCD sample* draws from the soft-$\mathbb{Z}_2$ ranges with $m_{12}^2 = 0$ and keeps a draw if both vacuum values are nonzero and the point is a minimum with exactly three Goldstone modes, until 1500 vacua or 40000 draws; it holds 1315 vacua, each scaled to $v = 246.22$ GeV.

### 2. Two-doublet model

In the two-doublet model, with the potential of Eq. (F4), two inequivalent minima coexist in 4 of the 150 softly broken $\mathbb{Z}_2$ potentials of the unbiased survey and in 4 of its 150 general ones, and the targeted scan finds 152 and 105 such pairs. Their asymmetry ranges over $|\varepsilon| = 7.4 \times 10^{-3}$–$0.74$ (soft $\mathbb{Z}_2$) and $2.7 \times 10^{-3}$–$1.1$ (general). At a representative soft-$\mathbb{Z}_2$ point, in units $|m_{11}^2| = 1$, the wrong-sign vacuum is a strict local minimum for $m_{12}^2 < 0.34$, with $\varepsilon$ linear in $m_{12}^2$ and a barrier that stays finite as $m_{12}^2 \to 0$. For an exchange-symmetric potential the two minima are exactly degenerate with $\varepsilon = 0$, and the one-loop quark Coleman–Weinberg term lifts them, by $\varepsilon = -1.9 \times 10^{-9}$ for $(y_u, y_d) = (0.01, 0.0002)$, for example. At that point the exact tree-level slope is $\varepsilon = -1.10\, m_{12}^2$, and the window $|\varepsilon| = 1.1$–$4.2 \times 10^{-14}$ needs $m_{12}^2 = -(0.96$–$3.9) \times 10^{-14}$ for $v' > v$, with the ordinary sector in the false vacuum, or $+(0.96$–$3.9) \times 10^{-14}$ for $v' < v$, with the mirror in the false vacuum. The barrier along the lowest path, through a saddle at which the relative phase of the doublets is $\pi/2$, is $89/234 \simeq 0.38$ of $|V_0|$, the depth of the vacua (the straight path to one of the gauge images of the vacuum gives 0.48, an upper bound). The minimal wall tension, along a neutral path through the relative CP phase, is $\sigma = 1.55$ in these units; the CP-even wall, $\sigma = 1.63$, is minimal among CP-conserving paths and a saddle of the full tension functional. The thin-wall exponent of the decay of the false vacuum is $\log_{10} S_E = 43.2$ at $|\varepsilon| = 10^{-14}$ and 43.1 to 41.3 across the window, so the false vacuum is stable; the thin-wall form holds by a wide margin.

At leading order in chiral perturbation theory [69–71] the QCD vacuum energy differs between $\bar\theta = \pi$ and 0 by $\Delta E = 2\Sigma_0 m_u = 9.7 \times 10^{-5}$ GeV$^4$ (band $8.7 \times 10^{-5}$ to $1.1 \times 10^{-4}$ GeV$^4$). Through $m_u \propto v_u$ and the heavy-quark thresholds of $\Lambda_{\rm QCD}$ it acts as a tadpole on the mirror sector at $\bar\theta' = \bar\theta + \pi$. At the representative point scaled to $v = 246.22$ GeV and $m_{12}^2 = 0$ it gives $\varepsilon_{\rm QCD} = -1.1 \times 10^{-13}$; over the 1315 stable vacua of the QCD sample the median $|\varepsilon_{\rm QCD}|$ is $1.1 \times 10^{-13}$, with 10–90% range $3.4 \times 10^{-14}$ to $7.1 \times 10^{-13}$. $\varepsilon = 0$ then needs $m_{12}^2/|m_{11}^2| = -9.6 \times 10^{-14}$ at that point, and the window needs $-1.06 \times 10^{-13}$ to $-1.34 \times 10^{-13}$ for $v' > v$ and $-8.6 \times 10^{-14}$ to $-5.7 \times 10^{-14}$ for $v' < v$. Since the shift exists only for a mirror at $\bar\theta' = \bar\theta + \pi$, whose neutron mass differs by tens of MeV (below), these values do not give an $n$–$n'$ window. With the QCD energy the thin-wall exponent is $\log_{10} S_E = 40.0$ at $m_{12}^2 = 0$; at the tuned window ends it is 41.9 to 44.5 for $v' > v$ and 40.9 to 42.0 for $v' < v$. The value 44.5, at the 3 $\mu$eV end with $v' > v$, is a central value: there the energy difference of the vacua is a small residue of a cancellation and changes sign over the band of $\varepsilon_{\rm QCD}$, so only $\log_{10} S_E \gtrsim 42$ is robust. $\Delta E$ is about $10^{-12}$ of the electroweak barrier and exists only below the QCD transition, so it does not select the vacua.

With the mirror at $\bar\theta' = \pi$, the leading-order vacuum rotates $u'$ by $\pi$ and conserves CP. Dashen's phenomenon [72] would need $m_d m_s/(m_d + m_s) < m_u < m_d m_s/(m_s - m_d)$, i.e. $m_u/m_d \in (0.952, 1.053)$ with the $\eta'$ integrated out; the physical $m_u/m_d = 0.46$ is below. With a finite $\eta'$ mass the CP-conserving vacuum at $\theta = \pi$ is stable if and only if $F^2 B m_u < [\chi_{\rm YM}^{-1} + (F^2 B m_d)^{-1} + (F^2 B m_s)^{-1}]^{-1}$, a condition on $F^2 B m_q/\chi_{\rm YM}$ rather than on $m_u/m_d$ alone, so the lower edge of the window depends on what is held fixed: 0.877 at fixed $B$ and $m_d$ (band 0.854 to 0.894) and 0.893 at fixed $m_u + m_d$ (band 0.872 to 0.908). The physical ratio lies below both. The mirror spectrum at leading order is $m'^2_{\pi^\pm} = B(m_d - m_u)$, $m'^2_{K^\pm} = B(m_s - m_u)$, $m'^2_{K^0} = B(m_d + m_s)$: $m'_{\pi^\pm} = 82$ MeV and $m'_{\pi^0} = 80$ MeV. The nucleons shift by $m_{n'} - m_n = -29$ MeV (band $-39$ to $-21$ MeV). Reference [9] quotes no mixing mass $\delta_m$; taking $\delta_m = (\Delta/2) \sin 2\theta$ at the corners of its window, unchanged in the mirror, the vacuum mixing lies in $\sin^2 2\theta = 4.7 \times 10^{-32}$ to $6.5 \times 10^{-30}$ over the four corners and the band of $m_{n'} - m_n$, so there is no observable oscillation.

In the soft-$\mathbb{Z}_2$ model with $\lambda_5 > 0$ a global neutral minimum can break CP spontaneously, for example with relative phase $\xi = 1.43$ ($\cos\xi = m_{12}^2/\lambda_5 v_1 v_2$), degenerate with its image $-\xi$. Then $\bar\theta = 3\xi$, and the bound $|\bar\theta| < 7.5 \times 10^{-11}$ from the neutron electric dipole moment [52, 53] needs $\xi \simeq 0$ or $2\pi/3$, i.e. $m_{12}^2 = -\lambda_5 v_1 v_2/2$ to a relative $4.3 \times 10^{-11}$; the QCD energy moves $\xi$ by only $2.7 \times 10^{-13}$.

### 3. SMM4b chains

The competitor search finds competing minima in at least 44 of the 240 quark types of chain N2 (41 of 180 in the near-$G_{54}$ chain, 0 of 10 for $U(3)^5$); all are strict local minima at 50 digits. Of its 50 mixed configurations of chain N2 (mirror in a competing minimum, ordinary sector in the global one), 24 are local minima of the 72-dimensional potential in every relative $S_3$ image and 28 in at least one; they have $\Delta E/E_{\rm cond} = 1.3\times10^{-4}$ to 0.26, $|\varepsilon_{\rm rms}| = 3.6\times10^{-4}$ to 0.12, and $|\varepsilon_Z| = 7\times10^{-5}$ to 0.09 with $I_D$, similar with $I_C$. The 48 mixed configurations of the near-$G_{54}$ chain have $|\varepsilon_{\rm rms}|/\epsilon_S = 0.0017$–0.84. In the scaling set, where the splitting directions are fixed, $|\varepsilon_{\rm rms}|$ and $\Delta E$ are linear in $\epsilon_S$, with $|\varepsilon_{\rm rms}| = (2.9\times10^{-3}$–$0.076)\,\epsilon_S$ and a generation spread (0.96–2.3) $\epsilon_S$; 27 of the 48 configurations lie inside this range. At the global vacua the exactly degenerate CKM images differ in $v_W$ by up to 0.02 (chain N2) and by at most $10^{-9}$ in the 45 class. No mixed configuration has $J \neq 0$: 59 of 98 have a degenerate same-type pair, and the others have $|J| \leq 2\times10^{-8}$, which counts as zero here: $J$ is taken as nonzero only if $|J|$ exceeds both $10^{-10}$ and ten times its spread under $10^{-8}$ relative perturbations of the couplings.

With $J$ computed from the CKM matrix of Newton-polished stationary points, exactly four vacua of the main sample have $J \neq 0$ in some of their six degenerate $S_3$ images: two in chain N2, with $J \in \{0, 2.0\times10^{-3}\}$ and $\{-4.3\times10^{-3}, 0\}$, all images local minima; in the near-$G_{54}$ chain, one with $J \in \{0, 9.2\times10^{-4}\}$, all images stable, and one with $J \in \{0, 1.7\times10^{-3}\}$ that is a saddle in every image. All other vacua have $|J| < 3\times10^{-14}$. The vacuum-value asymmetry between the $J' = 0$ and $J \neq 0$ images is $|\varepsilon_W| = 1.2\times10^{-5}$–$2.1\times10^{-3}$ with $I_D$ for the three stable vacua, and $6.8\times10^{-7}$–$2.1\times10^{-3}$ with the saddle included; the scheme dependence is of order one, with the same sign for $I_D$, $I_C$ and $I_{PS}$. The CP images of these vacua are not degenerate, $\Delta E/E_{\rm cond} = 0.07$–1.7.

The explicit breaking that would lift the axion of Sec. V E was tested on the 45-class local minima of the main sample (39 of chain N2 without a massless quark, 28 of the near-$G_{54}$ chain). In 27 of 39 and 22 of 28 vacua a diagonal $\mathbb{Z}_3 \subset S_3 \times \mathbb{Z}_3^A$ survives, $\langle O_{quqd}\rangle = 0$, and the lift is of third order (a potential $\cos(3a/f)$ in the axion field $a$). For axion masses of 10 and 100 GeV the breaking amplitude must be $1.4\times10^5$–$1.5\times10^6$ and $1.4\times10^7$–$1.5\times10^8$ GeV$^4$ in chain N2, while $|\bar\theta| < 7.5\times10^{-11}$ allows at most $8.5\times10^{-15}$ GeV$^4$ ($q = 1$) or $2.5\times10^{-15}$ GeV$^4$ ($q = 3$). A uniformly distributed $\bar\theta$ satisfies the bound with probability $2.4\times10^{-11}$. QCD instantons at the condensation scale contribute only about $10^{-14}$ of the topological susceptibility. At the 10 GeV benchmark with $O_{quqd}$ the degenerate pair splits by $2.5\times10^{-6}$ to $3\times10^{-2}$ of the heaviest quark mass of the vacuum, the vacua stay aligned, $|\delta \ln v| \leq 1.5\times10^{-3}$, and $J \neq 0$ appears in 29 vacua of chain N2 and 22 of the near-$G_{54}$ chain, up to $|J| = 0.069$. Near $G_{54}$ the light 36 and heavy 10 persist in 25 of 28 vacua. With real four-fermion couplings and real $c$, $\bar\theta \in \{0, \pi\}$ in most 45 vacua, and the others break CP spontaneously.


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